Electric potential of angled plates

Abstract

An application of conformal maps.

Index Terms

Integral, Statistics

This is the first of a series of posts on solving Laplace equation using conformal maps. I have a recent post on conformal maps that introduces the conformal map. This post will focus on a simple geometry to provide an easy, pedagogical example. We will later tackle fancy problems.

I. Two plates at an angle

Consider two semi-infinite plates places at an angle θ_s separated by an insulator of negligible size at the origin as illustrated in Fig. 1. The horizontal plate is grounded and the other one is held at a potential φ_s.

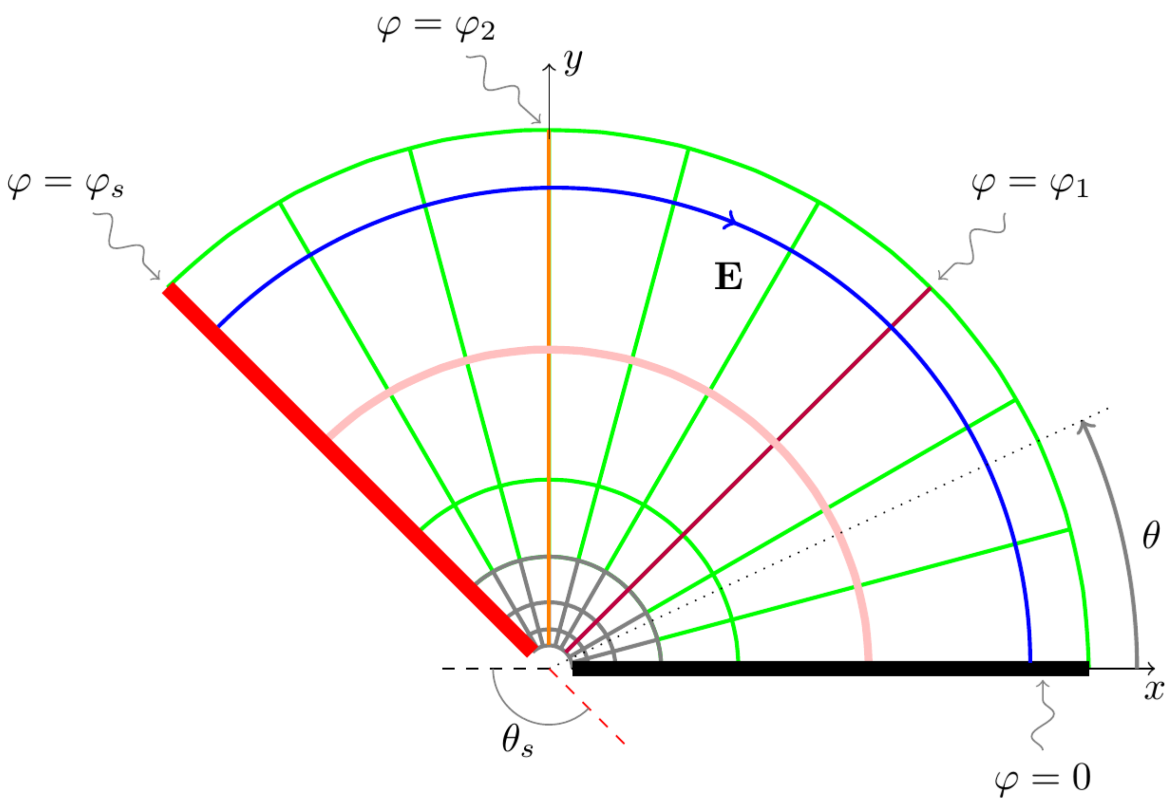

Figure 1: An illustration of electric field and the potential created by two plates held at different voltages.

The electric potential increases along the angular direction θ, creating the arcs of circles in Fig. 1. The electric field will be orthogonal to the these arcs. The color coding in Fig. 1 will be relevant when we discuss the conformal maps.

We need to solve the Laplace equation in the cylindirical coordinates in two dimensions:

email: quarktetra@gmail.com
Find the interactive HTML-document here.

$$\vec{\nabla}^2\varphi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi = \frac{\partial^2}{\partial\theta^2}\varphi = 0, \tag{1}$$

where the term with r derivatives drop since there is no r dependence due to the symmetry. It is easy to solve Eq. (1):

$$\varphi(\theta) = \varphi_s\frac{\theta}{\theta_s}, \quad 0 \le \theta \le \theta_s, \tag{2}$$

with the corresponding electric field:

$$\mathbf{E} = -\nabla\varphi(\theta) = -\frac{\varphi_s}{\theta_s r}\hat{\theta}. \tag{3}$$

We are showing the solution for $0 \le \theta \le \theta_s$. In the complementary domain of angle, $\theta_s \le \theta < 2\pi$, the solution is similar.

II. Conformal mapping

The boundary conditions imposed by the plates at $\theta = 0$ and $\theta = \theta_s$ forced us to solve the Laplace equation in curvilinear coordinates. If we could map them on to Cartesian boundaries in another domain, the differential equation would have been easier. The electric fields form the arcs of circl, and they are perpendicular to the equipotential lines, whicha re radial. We want to resolve the problem in the context of complex variables using a conformal map. We want to find the function $\theta(x,y)$ which satisfies Eq. (1). We want to map the circles and the radial rays into a simpler view in the mapped space. The circles in the complex plane are represented by $z = re^{i\theta}$. As it has a built-in exponential, intuitively we can see that we can undo that if we tried $f(z) = \ln(z)$ as the mapping function.

$$f(z) = \ln(z) = \ln(r) + i\theta \equiv u + iv. \tag{4}$$

This maps (x,y) to (u,v), and the potential function φ to ϕ. Furthermore the harmonic feature of φ with respect to (x,y), i.e, Eq. (1), is still valid for ϕ with respect to (u,v), see my earlier post. Figure 2 shows that circles on the right are mapped to vertical lines in the (u,v) space, whereas the radial lines are mapped to the horizontal lines. The plots are color coded to make the mapping easier to follow.

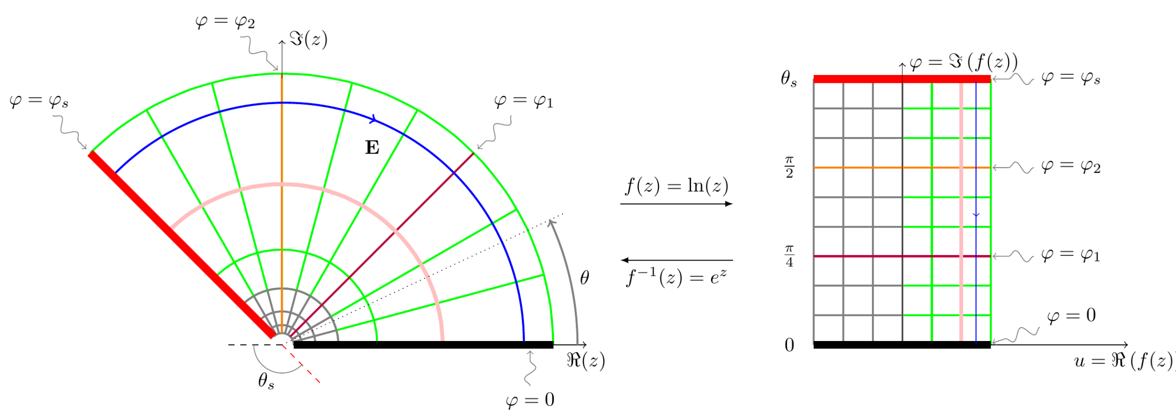

Figure 2: The conformal map $\ln(z)$ converts circles and radial lines to straight lines. The boundary plates are mapped to Cartesian boundaries.

The Laplace equation for ϕ in the (u,v) space reads:

$$(\partial_u^2 + \partial_v^2)\phi = 0. \tag{5}$$

The boundary conditions at $\theta = 0$ and $\theta = \theta_s$ are mapped to $v = 0$ and $v = \theta_s$. With these boundary conditions, and realizing that the potential needs to be constant along the v axis, the Laplace equation reduces to

$$\partial_v^2 \phi(v) = 0 \implies \phi(v) = \varphi_s \frac{v}{\theta_s}. \tag{6}$$

Finally, we revert v to (r, θ) coordinates using Eq. (4): $v = \theta$, which implies

$$\varphi(\theta) = \varphi_s \frac{\theta}{\theta_s}. \tag{7}$$

We can take this and calcualte the corresponding electric field by computing the gradient, however, we can get fancy and use the concept of complex potential.

III. Complex potential

Given the real valued potential function ϕ, we can analytically complement it to create a complex potential:

$$\Phi = \phi + i\psi, \tag{8}$$

where constant ϕ's are the equipotential curves, and constant ψ's are the streamlines. The components of $\Phi(z)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial \phi}{\partial u} = \frac{\partial \psi}{\partial v} \text{ and } \frac{\partial \phi}{\partial v} = -\frac{\partial \psi}{\partial u}. \tag{9}$$

We can integrate for ψ to get:

$$\psi = -\int du \frac{\partial \phi}{\partial v} = -\varphi_s \frac{u}{\theta_s} = -\varphi_s \frac{\ln(r)}{\theta_s}. \tag{10}$$

Putting this back we get

$$\Phi = \frac{\varphi_s}{\theta_s}(v - iu) = -i\frac{\varphi_s}{\theta_s}(u + iv) = -i\frac{\varphi_s}{\theta_s}(\ln(r) + i\theta) = -i\frac{\varphi_s}{\theta_s}\ln(z). \tag{11}$$

The gradients of ϕ and ψ are perpendicular to each other:

$$(\phi_x, \phi_y)^T \cdot (\psi_x, \psi_y) = 0, \tag{12}$$

which means the level lines are perpendicular too. We also define a vector quantity F as the gradient of ϕ:

$$\mathbf{F} = \vec{\nabla}\phi = (\phi_x, \phi_y). \tag{13}$$

The object $\vec{F}$ is divergence and curl free:

$$\begin{aligned} \vec{\nabla} \times \mathbf{F} &= \phi_{yx} - \phi_{xy} = 0, \\ \vec{\nabla} \cdot \mathbf{F} &= \phi_{xx} + \phi_{yy} = 0. \end{aligned} \tag{14}$$

This makes $\vec{F}$ a compatible vector field for incompressible and irrotational flows and electrostatic problems.