L^2 is a Hermitian operator

Abstract

A proof that square of the angular momentum vector is a Hermitian operator.

Index Terms

angular momentum, operator algebra

We need to show that

$$\int \psi_1^*(L^2\psi_2)d\Omega = \left[\int \psi_2^*(L^2\psi_1)d\Omega\right]^*, \tag{1}$$

where

$$L^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{2}$$

The second part of L^2 operator is easier to handle. The relevant part of the integral is the ϕ integral, which can be computed as follows

$$\begin{aligned}
\int_0^{2\pi}\psi_1^*\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_2\right]d\phi &= \int_0^{2\pi}\frac{\partial}{\partial\phi}\left[\psi_1^*\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\psi_2\right]d\phi - \int_0^{2\pi}\left(\frac{\partial}{\partial\phi}\psi_1\right)^*\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\psi_2\right)d\phi \\
&= -\int_0^{2\pi}\frac{\partial}{\partial\phi}\left[\left(\frac{\partial}{\partial\phi}\psi_1\right)^*\frac{1}{\sin^2\theta}\psi_2\right]d\phi + \int_0^{2\pi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_1\right)^*\psi_2 d\phi \qquad (3) \\
&= \left[\int_0^{2\pi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_1\right)\psi_2^* d\phi\right]^*, \qquad (4)
\end{aligned}$$

where we dropped the first terms in the first two lines as the they are the difference of the integrand at $\phi = 2\pi$ and $\phi = 0$, and that is zero as ϕ coordinate is 2π periodic.

The first part of L^2 operator seems to be harder because when we integrate by parts $\frac{\partial}{\partial\theta}$ will act on $\sin\theta$, which will complicate the problem. However, we can avoid it by a change of variable $u = \cos\theta$. The relevant part of the integral is the $dcos\theta$ integral, and with the above transformation it becomes,

$$\begin{aligned}
\int_0^{\pi}\psi_1^*\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi_2\right)d\cos\theta &= \int_{-1}^{1}\psi_1^*\frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_2\right)du \\
&= -\int_{-1}^{1}\frac{\partial}{\partial u}\psi_1^*\left((1-u^2)\frac{\partial}{\partial u}\psi_2\right)du \\
&= -\int_{-1}^{1}\left((1-u^2)\frac{\partial}{\partial u}\psi_1^*\right)\frac{\partial}{\partial u}\psi_2 du \\
&= \int_{-1}^{1}\frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_1^*\right)\psi_2 du \\
&= \left[\int_{-1}^{1}\psi_2^*\frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_1\right)du\right]^* \\
&= \left[\int_0^{\pi}\psi_2^*\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi_1\right)d\cos\theta\right]^*, \qquad (5)
\end{aligned}$$

where we dropped again some surface terms as $u^2 - 1 = 0$ at $u = \pm 1$. (If you prefer $\sin\theta d\theta$ integral instead of $dcos\theta$ integral, you will not need to change the variable.) This completes the proof that L^2 is Hermitian.

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