Abrikosov-Nielsen-Olesen flux tubes

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Abstract

A quick derivation of ANO vortices in the context of spontaneous symmetry breaking.

Contents

Topological defects are remnants of spontaneously broken local or global symmetries. They appear in many fields of physics ranging from high energy physics to solid state physics. One of the most well known topological defects appears in magnetic materials. Let us consider a material which is composed of clusters with magnetic moments. The dynamics of the system can be described by a Heisenberg type Hamiltonian, which is invariant under rotations, i.e. there are no preferred directions for the system. However, the physical realization of the ground state of the system is not rotationally invariant. The direction of the magnetic moments are chosen randomly at different locations. Nearby moments align with each other and create a domain structure. The magnetization smoothly interpolates between different domains, and the width of the transition range is the thickness of the domain wall. The domain wall is the topological defect that emerges upon breaking of the rotational symmetry of the system by randomly chosen magnetization. This is an example of spontaneously broken *global* symmetry. The domain walls are physical objects: they carry (magnetic) energy, and they can be moved or rotated by external currents or magnetic fields.

An example of spontaneously broken local symmetry occurs in superconducting materials. If a superconducting material is placed in a strong magnetic field, the magnetic field penetrates into the material at certain locations at which the superconductivity is lost. The magnetic field forms flux tubes which are one dimensional topological defects known as Abrikosov-Nielsen-Olesen flux tubes [1]. Abrikosov-Nielsen-Olesen flux tubes are topological defects associated with spontaneously broken $U(1)$ gauge symmetry. For the case of high energy physics, vortices or strings may form as a result of spontaneously broken unified theories. In the following sections we first outline the field theoretical background of formation of topological defects. In the first part of the thesis, we focus on vortices. We consider normalization of the mass and central charge of vortices in $\mathcal{N} = 2$ supersymmetric field theory. In the second part of the thesis, we consider strings which can be constructed as vortices extended along an additional dimension. We then discuss the SBGW due cusps and kinks on cosmic strings.

1 Formation of Topological Defects

Topological defects are relics of spontaneously broken symmetries. The exact nature of the defect depends on the group of the symmetry broken. Below we consider two important cases.

1.1 Spontaneously Broken Global Symmetries

Let us consider the Lagrangian for a complex scalar field:

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - V(\varphi, \varphi^*). \tag{1}$$

The potential can be chosen to be of the form

$$V(\varphi, \varphi^*) = \frac{\lambda^2}{2}\left(|\varphi|^2 - \frac{\eta^2}{2}\right)^2, \tag{2}$$

which is shown in Fig. 1.

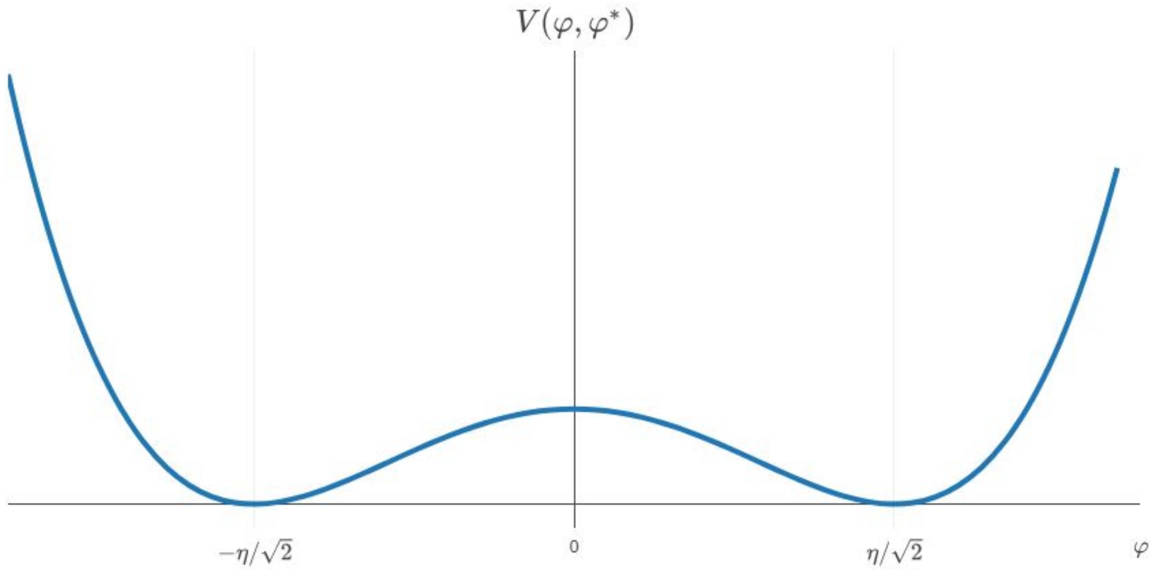

Figure 1: The quartic scalar potential.

The Lagrangian in Eq. (1) has a global $U(1)$ symmetry, i.e. it remains invariant under the phase rotations:

$$\varphi \to e^{i\theta}\varphi, \tag{3}$$

where θ is a constant real number. Although the field theory defined by the Lagrangian in Eq. (1) is invariant under the phase rotations, the vacuum state of the field is not. The vacuum state solution is given by the field configuration that minimizes the potential, which is

$$\varphi_V = \frac{\eta}{\sqrt{2}}e^{i\theta_V}. \tag{4}$$

θ_V is the phase of field at the vacuum state, which has no physical significance since it can be removed by a $U(1)$ rotation. The solution φ_V is clearly not invariant under $U(1)$
rotation, hence the $U(1)$ symmetry is spontaneously broken. The results of the broken symmetry can be seen by expanding the field around the vacuum solution. It is convenient to separate out the radial and angular components of the field by using the following expansion [2]

$$\varphi = \frac{\eta + \xi}{\sqrt{2}}e^{i\alpha}, \tag{5}$$

where θ_V is set to zero. Plugging this expansion to Eq. (1) we get

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \frac{\lambda^2\eta^2}{2}\xi^2 + \frac{\eta^2}{2}\partial_\mu\alpha\partial^\mu\alpha + \text{interaction terms}. \tag{6}$$

The first two terms in the effective Lagrangian in Eq. (6) represent a neutral particle ξ which has mass $\lambda\eta$. It is important to note that ξ corresponds to the radial excitation in the potential well, therefore the particle *sees* the curvature of the potential. On the other hand, the field α has no mass term in Eq. (6). It corresponds to the angular excitation in the Mexican hat-shaped potential. This massless mode is referred to as the *Goldstone Boson*. Whenever a global symmetry is spontaneously broken Goldstone Bosons which correspond to the excitation of the fields along the flat directions of the potential are generated. On the other hand, if the broken symmetry is a local symmetry, the degrees of freedom of the excitations along the flat directions are absorbed into the longitudinal component of gauge bosons which acquire mass upon spontaneously breaking the symmetry. This mechanism is crucial for the formation of vortices and flux tubes and hence it is discussed in detail below.

1.2 Spontaneously Broken Local Symmetries

In order to make the global $U(1)$ symmetry defined in Eq. (3) local, one introduces a gauge field A_μ with the following transformation

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \theta(x), \tag{7}$$

where e is the coupling constant. The partial derivatives are replaced with the gauge covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu + ieA_\mu. \tag{8}$$

With these definitions, the local $U(1)$ invariant Lagrangian can be written as

$$\mathcal{L} = \mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi^* - V(\varphi,\varphi^*) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10}$$

is the field strength of the gauge field. The vacuum solutions are still as given in Eq. (4) and we can use the expansion in Eq. (5). With this expansion the kinetic term in Eq. (9) can be written as

$$\begin{aligned} \mathcal{L}_K &= \mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi^* \\ &= (\partial_\mu + ieA_\mu)(\frac{\eta+\xi}{\sqrt{2}}e^{i\alpha})(\partial^\mu - ieA^\mu)(\frac{\eta+\xi}{\sqrt{2}}e^{-i\alpha}) \\ &= \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}(\partial_\mu\alpha + eA_\mu)(\partial^\mu\alpha + eA^\mu)(\eta+\xi)^2. \end{aligned} \tag{11}$$

We note that $\partial_\mu\alpha$ term can be absorbed into A_μ by gauging as described in Eq. (7), which shows that the would-be Goldstone boson is absorbed into the longitudinal component of the vector field. The full Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \frac{\lambda^2\eta^2}{2}\xi^2 + \frac{1}{2}\eta^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{interaction terms}, \tag{12}$$

which describes an interacting theory with a massive scalar and massive vector field. The number of degrees of freedom before and after the symmetry breaking is the same: one degree of freedom from the complex field is transferred to the vector field, which becomes massive, and hence it can have longitudinal polarization.

2 Vortices and Strings

In this section we reproduce the vortex solutions for a spontaneously broken local Abelian symmetry in $2+1$ dimensions. The Lagrangian for a complex scalar field coupled to the gauge field is given by

$$\mathcal{L} = \mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi^* - \frac{e^2}{2}\left(|\varphi|^2 - \eta^2\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{13}$$

We would like to consider the static solutions in $2+1$ dimensions. Eliminating the terms with time derivatives we can express the energy density as

$$\mathcal{E} = -\mathcal{L} = \mathcal{D}_k\varphi\mathcal{D}^k\varphi^* + \frac{e^2}{2}\left(|\varphi|^2 - \eta^2\right)^2 + \frac{1}{2}F_{k\,l}F^{k\,l}, \tag{14}$$

where $k = 1, 2$ denotes the space indices. It is important to note that the potential chosen here is a special case of Eq. (2), where λ is set to e , which is the the coupling constant (and also note that η is re-scaled by a factor $\sqrt{2}$). In this special case the equations of motion, which are a priori second order differential equations, can be reduced to first order differential equations by Bogomol'nyi completion [3]. The energy density given in Eq. (14) can be written in the following form

$$\mathcal{E} \;\; = \;\; \frac{1}{2}\left|(\mathcal{D}_k + i\epsilon_{kl}\mathcal{D}_l)\varphi\right|^2 + \frac{1}{2}\left(F_{12} + e(|\varphi|^2 - \eta^2)\right)^2 + \epsilon_{kl}\partial_l(e\eta^2 A_l - i\varphi^*\mathcal{D}_l\varphi), \tag{15}$$

where ϵ_{kl} is the two dimensional Levi-Civita tensor with the convention $\epsilon_{12} = 1$. The first two terms in this equation are positive definite and the last one is a boundary term. Therefore the energy can be minimized if the following equations are satisfied:

$$\begin{aligned} (\mathcal{D}_1 \pm i\mathcal{D}_2)\varphi &= 0, \\ F_{12} + e(|\varphi|^2 - \eta^2) &= 0, \end{aligned} \tag{16}$$

which are first order Bogomol'nyi equations. The last term in Eq. (15) is a surface term. If one integrates the last term over space coordinates, the result reads

$$\mathcal{Z} \;\; \equiv \;\; \int d^2x\epsilon_{kl}\partial_l(e\eta^2 A_l - i\varphi^*\mathcal{D}_l\varphi) = e\eta^2\int_{r\to\infty} rd\theta A_\theta, \tag{17}$$

which is proportional to the winding number of the gauge field (The second term in Eq. 17 vanishes exponentially.) $\mathcal{Z}$ is referred to as the *central charge*, since it commutes with the generators of the supersymmetric extension of the model (To be more precise, for the case of vortex, $\mathcal{Z}$ commutes with a portion of the supersymmetry generators. The asymptotic solutions of the Bogomol'nyi equations are

$$\begin{aligned} \varphi &= \eta e^{in\theta}, \\ A_k &= -n\frac{\epsilon_{kl}x^l}{r^2}, \end{aligned} \tag{18}$$

where n is an integer that represents the winding number. The magnetic field corresponding to the vector potential is F_{12} and it is confined to a region with radius of scale $1/\eta$. This is again in agreement with the conclusion that the gauge boson acquires a mass of η , and therefore the interaction strength decays exponentially with the distance.

The mass of vortex configuration reads

$$\mathcal{M} \;\; \equiv \;\; \int d^2x\mathcal{E} = \mathcal{Z} = 2\pi\eta^2|n|. \tag{19}$$

The equality of the mass and central charge is called the Bogomol'nyi Prasad Sommerfield (BPS) saturation. The BPS saturation is far from coincidence: it holds even under quantum corrections in supersymmetric extensions. If the vortex configuration is extended along the z -axis, the result is a flux tube. The mass per unit length of the tube can be described as the tension, and Eq. (19) shows that the tension is proportional to the square of η , which is the energy scale of the phase transition. Therefore the tension of the string critically depends on the energy scale of the symmetry breaking.

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[3] E. B. Bogomol'nyi, "The stability of classical solutions," *Sov. J. Nucl. Phys. (Engl. Transl.); (United States)*, vol. 24:4, Oct. 1976 [Online]. Available: https://www.osti.gov/biblio/7309001