Potentials of a split cylindrical shell

Abstract

Double dipping with conformal maps!

Index Terms

Integral, Statistics

This is the second one of a series of posts on solving Laplace equation using conformal maps. I have a recent introductory post on conformal maps. We will tackle a problem which is a bit harder the previous warm up exercise. In fact, we are going to double down and apply comformal maps twice!

I. The setup

Consider a thin, long, conducting cylinder of unit radius split into two equal pieces. The pieces are separated by a small gap and held at two different voltages. Figure 1 shows the cross section of the setup. The goal is to calculate the electric potential and the electric field.

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Find the interactive HTML-document here.

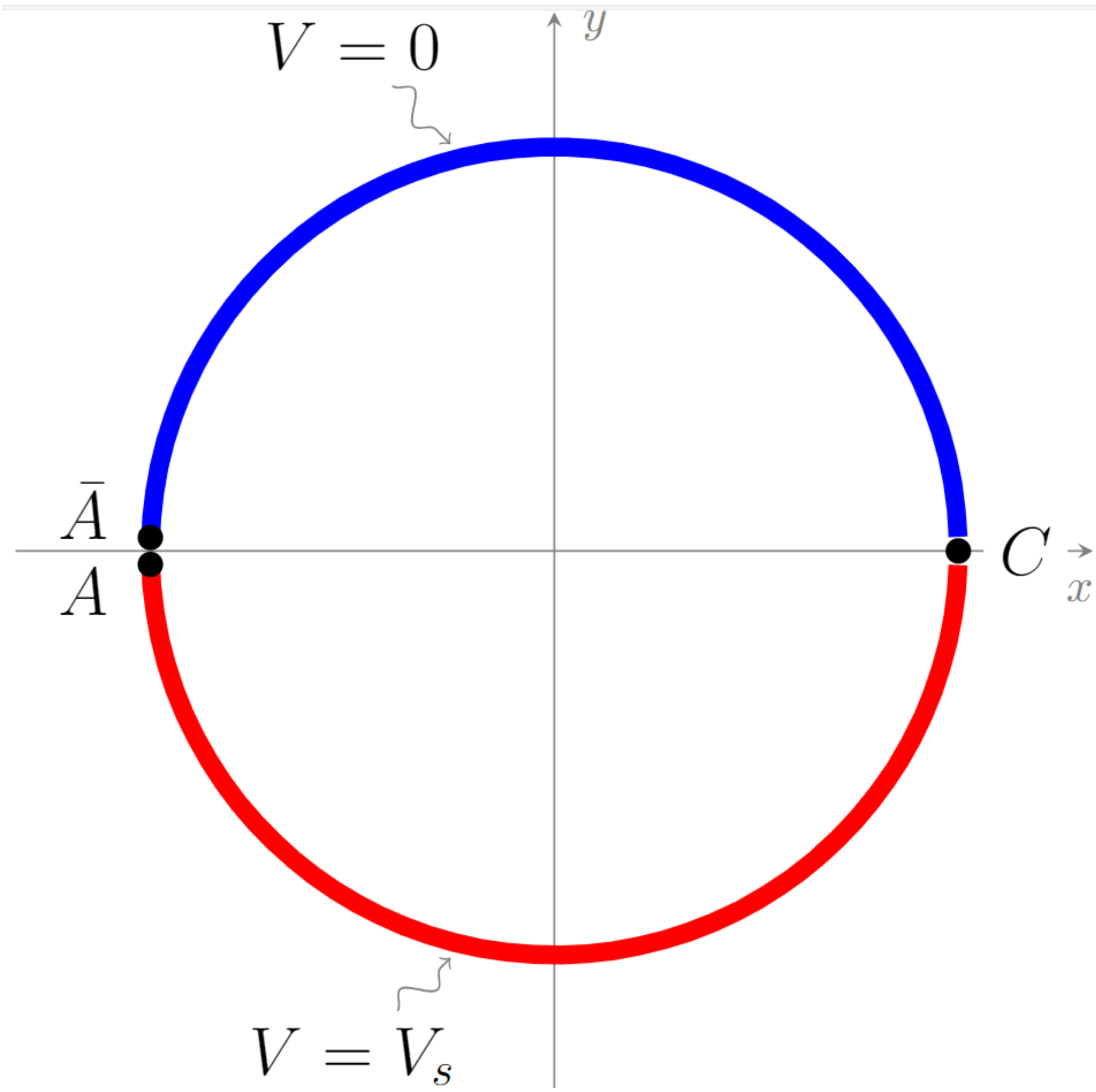

Figure 1: The crosssection of a cylindrical shell split into two pieces.

II. Old school solution

We need to solve the Laplace equation in the cylindrical coordinates in two dimensions:

$$\vec{\nabla}^2\varphi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi = 0, \tag{1}$$

The boundary conditions are imposed at $r = 1$:

$$\varphi(1,\theta) = \begin{cases} 0, & 0 < \theta < \pi, \\ V_s, & \pi < \theta < 2\pi, \end{cases} \tag{2}$$

where the angle θ is measured from the positive real axis. We can separate the variables as $\varphi(r,\theta) = F(r)G(\theta)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)G + \frac{F}{r^2}\frac{d^2G}{d\theta^2} = 0,, \tag{3}$$

or equivalently

$$\frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = -\frac{1}{G}\frac{d^2G}{d\theta^2}. \tag{4}$$

Since the left-hand side of Eq. (4) depends on r only, and the right one depends on θ only, overall they can only be equal to a constant, which we will call ν^2. The partial differential equation is then separated into two ordinary differential equations:

$$r\frac{d}{dr}\left(r\frac{dF}{dr}\right) = \nu^2 F, \text{ and } \frac{d^2G}{d\theta^2} = -\nu^2 G. \tag{5}$$

$\nu = 0$ case requires a bit of special treatment: A constant or $\ln(r)$ will work for F, and a first order polynomial, $a + b\,\theta$, will work for G. However, we will want our solution to not blow up at $r = 0$, and therefore we can't include $\ln(r)$. Furthermore, since θ is the angle, to preserve the periodicity, we can't have the $b\,\theta$ term. For $\nu \neq 0$, $F \propto r^{\pm\nu}$ and $G \propto e^{\pm i\nu\theta}$ will solve the equations. But, again, the requirement of a finite solution at $r = 0$ will eliminate $r^{-\nu}$ solutions inside the cylinder. Finally, the periodicity will require ν to be an integer. As the equations are linear, we can include them with arbitrary coefficients:

$$\varphi(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n e^{in\theta} + b_n e^{-in\theta}\right). \tag{6}$$

We can now impose the boundary condition in Eq. (2):

$$\varphi(1,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n e^{in\theta} + b_n e^{-in\theta}\right). \tag{7}$$

We can isolate the first Fourier coefficient, a_0:

$$\int_0^{2\pi} d\theta \varphi(1,\theta) = 2\pi a_0 + \underbrace{\sum_{n=1}^{\infty}\int_0^{2\pi} d\theta \left(a_n e^{in\theta} + b_n e^{-in\theta}\right)}_{\to\, 0} \implies a_0 = \frac{V_s}{2}, \tag{8}$$

which is simply the average value of the voltage across the disk surface. Similarly

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}\varphi(1,\theta) = \frac{V_s}{2\pi}\int_\pi^{2\pi} d\theta e^{-in\theta} = \frac{iV_s}{2\pi n}\left(1 - e^{-in\pi}\right) = \frac{iV_s}{2\pi n}\left(1 - (-1)^n\right), \tag{9}$$

and

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{in\theta}\varphi(1,\theta) = \frac{V_s}{2\pi}\int_\pi^{2\pi} d\theta e^{in\theta} = \frac{-iV_s}{2\pi n}\left(1 - e^{in\pi}\right) = \frac{-iV_s}{2\pi n}\left(1 - (-1)^n\right), \tag{10}$$

which shows that only odd n terms will contribute. To make this more explicit, let's define $n = 2m + 1$ where $m = 1, 2, \cdots$ to get:

$$\begin{aligned}
\varphi(r,\theta) &= \frac{V_s}{2} + \frac{V_s}{\pi}\sum_{m=1}^{\infty} r^{2m+1}\left(\frac{i}{2m+1}e^{i(2m+1)\theta} + \text{complex conjugate}\right) \\
&= \frac{V_s}{2} + \frac{2V_s}{\pi}\Re\sum_{m=1}^{\infty}\left(\frac{i}{2m+1}\left(re^{i\theta}\right)^{2m+1}\right).
\end{aligned} \tag{11}$$

The summation coefficients should remind you of arctan, but we need to massage it a bit. Let's define $\theta = \tilde{\theta} + \pi/2$ and take a look at the argument of the summation:

$$\begin{aligned}
\frac{i}{2m+1}\left(re^{i\theta}\right)^{2m+1} &= \frac{i}{2m+1}\left(re^{i\tilde{\theta}+i\pi/2}\right)^{2m+1} = \frac{i}{2m+1}\left(re^{i\tilde{\theta}}\right)^{2m+1}\left(e^{i\pi}\right)^m\left(e^{i\pi}\right)^{1/2} \\
&= -\frac{(-1)^m}{2m+1}\left(re^{i\tilde{\theta}}\right)^{2m+1} = -\frac{(-1)^m}{2m+1}\left(re^{i(\theta-\pi/2)}\right)^{2m+1}.
\end{aligned} \tag{12}$$

Putting this back in Eq. (11) we get:

$$\varphi(r,\theta) \quad = \quad \frac{V_s}{2} - \frac{2V_s}{\pi}\Re\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2m+1}\left(re^{i(\theta-\pi/2)}\right)^{2m+1}\right) = \frac{V_s}{2} - \frac{2V_s}{\pi}\Re\left(\arctan\left(re^{i(\theta-\pi/2)}\right)\right). \quad (13)$$

Now we have to push $\Re$ through arctan into the argument. For this, it is convenient to work in a different representation of arctan. Consider the integral below:

$$\int dz\frac{1}{1+z^2} = \int d\tan\alpha\frac{1}{1+\tan^2\alpha} = \int d\alpha\sec^2\alpha\cos^2\alpha = \int d\alpha = \alpha + C = \arctan(z) + C, \quad (14)$$

where we defined $z = \tan\alpha$ and reverted back at the end of the computation. We can also compute the integral using the partial fractions method:

$$\int dz\frac{1}{1+z^2} = \frac{i}{2}\int dz\left(\frac{1}{i+z} + \frac{1}{i-z}\right) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) + C, \quad (15)$$

where C is a constant that will set the domain of the angle as well as the branch cut of ln. Since Eqs.(14) and (15) are equivalent, we have:

$$\arctan(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right). \quad (16)$$

Let's now take $z = x + iy$, and push $\Re$ through arctan:

$$\Re\left(\arctan z\right) \quad = \quad \Re\left(\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)\right) = \frac{1}{2}\Im\left(\ln\left(\frac{i+z}{i-z}\right)\right) = \frac{1}{2}\left(\arctan\left(\frac{1+y}{x}\right) + \arctan\left(\frac{1-y}{x}\right)\right)(17)$$

We now need to figure out how to add the arctan's on the right. Let's derive that quickly. Let's define angles α and β with $\tan\alpha = a$ and $\tan\beta = b$. We can use the formula for tangent of sums

$$\begin{aligned}\tan(\alpha+\beta) \quad &= \quad \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{a+b}{1-ab} \\ &\Longrightarrow \quad \arctan\tan(\alpha+\beta) = \alpha+\beta = \arctan a + \arctan b = \arctan\frac{a+b}{1-ab}.\end{aligned} \quad (18)$$

In our case, from Eq. (17), we have $a = \frac{1+y}{x}$ and $b = \frac{1-y}{x}$, which gives:

$$\Re\left(\arctan z\right) \quad = \quad = \frac{1}{2}\arctan\left(\frac{\frac{2}{x}}{1-\frac{1-y^2}{x^2}}\right) = \frac{1}{2}\arctan\left(\frac{2x}{1-x^2-y^2}\right) = \frac{1}{2}\arctan\left(\frac{2\Re(z)}{1-|z|^2}\right). \quad (19)$$

Inserting this back into Eq. (13), we get:

$$\begin{aligned}\varphi(r,\theta) \quad &= \quad \frac{V_s}{2} - \frac{V_s}{\pi}\arctan\left(\frac{2r\cos(\theta-\pi/2)}{1-r^2}\right) = \frac{V_s}{2} - \frac{V_s}{\pi}\arctan\left(\frac{2r\sin(\theta)}{1-r^2}\right) \\ &= \quad \frac{V_s}{2} - \frac{V_s}{\pi}\left(\pi/2 - \operatorname{arccot}\left(\frac{2r\sin(\theta)}{1-r^2}\right)\right) = \frac{V_s}{\pi}\arctan\left(\frac{1-r^2}{2r\sin(\theta)}\right),\end{aligned} \quad (20)$$

which is the final answer for the electric potential inside the cylinder!

III. Mapping circles and lines

The boundary of the original problem consists of arcs of circle. We want to map those boundaries to Cartesian ones. A linear fractional transformation can accomplish this goal:

$$\omega = i\frac{1-z}{1+z}. \quad (21)$$

We can quickly verify what how the upper arc, $z = e^{i\theta}$, for $0 < \theta < \pi$ gets mapped.

$$\omega = i\frac{1-e^{i\theta}}{1+e^{i\theta}} = i\frac{e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right)}{e^{i\theta/2}\left(e^{i\theta/2} + e^{i\theta/2}\right)} = \tan(\theta/2), \quad (22)$$

which is the positive real axis on the ω-plane. Similarly, the lower arc, $z = e^{-i\theta}$, for $0 < \theta < \pi$ gets mapped as:

$$\omega = i\frac{1 - e^{-i\theta}}{1 + e^{-i\theta}} = i\frac{e^{-i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)}{e^{-i\theta/2}\left(e^{-i\theta/2} + e^{-i\theta/2}\right)} = -\tan(\theta/2), \tag{23}$$

which is the negative real axis. The full mapping is shown in Fig. 2.

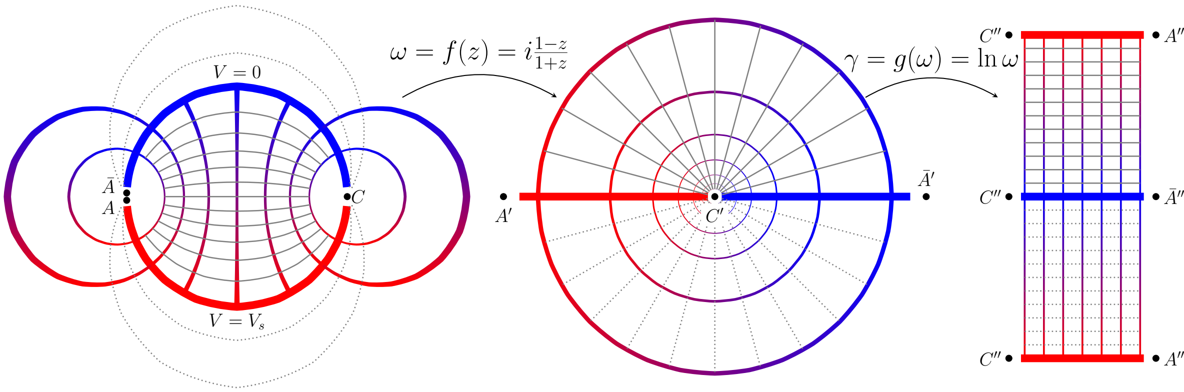

Figure 2: The first transformation maps the boundaries to the x axis. Equipotential lines are radial rays and electric field lines are arcs of circle. The second transformation maps the lines onto the nice and clean Cartesian grid. The original boundary plates are mapped to the horizontal lines at 0 and π in the γ domain.

The fractional mapping in Eq. (21) moves the boundaries such that it becomes relatively easy to solve the Laplace equation in cylindrical coordinates, see the earlier post. However, no one is stopping us from doubling down and moving from ω plane to a new γ plane:

$$\gamma = \ln \omega \equiv \tilde{u} + i\tilde{v}. \tag{24}$$

It is convenient to use the polar parameterization in the ω plane, i.e., $\omega = \rho e^{i\alpha}$, and revisit Eq. (24):

$$\gamma = \ln \omega = \ln \rho + i\alpha = \tilde{u} + i\tilde{v}. \tag{25}$$

As we are jumping from z to ω, and from ω to γ, the electric potential transforms due to its argument. You will find that it is pretty confusing already, and textbooks[1] add to this confusion by using the same name for those functions. Let's be very precise and clear up this mess once and for all. φ is a function of the original variables, (x, y) in Cartesian or (r, θ) in cylindrical coordinates. We represent the original pair of variables as a complex parameter $z = x + iy$ or $z = re^{i\theta}$. So, φ is a function of z: $\varphi(z)$. Now z gets mapped to ω by a function $\omega = f(z) = i\frac{1-z}{1+z} \equiv u + iv \equiv \rho e^{i\alpha}$, which can be reverted as $z = f^{-1}(\omega)$. Inserting this back in $\varphi(z)$, we get $\varphi(z) = \varphi\left(f^{-1}(\omega)\right) = \left(\varphi\, o\, f^{-1}\right)(\omega) \equiv \phi(\omega)$.

ω plane gets mapped to $\gamma = g(\omega) = \ln \omega \equiv \tilde{u} + i\tilde{v}$, which can be inverted as $\omega = g^{-1}(\gamma)$. Putting this back in again will give: $\phi(\omega) = \phi(g^{-1}(\gamma)) = \left(\phi\, o\, g^{-1}\right)(\gamma) \equiv \tilde{\phi}(\gamma)$. To summarize, we have the following functions:

$$\phi = \varphi\, o\, f^{-1}, \text{ and } \tilde{\phi} = \phi\, o\, g^{-1} = \varphi\, o\, f^{-1}\, o\, g^{-1} \tag{26}$$

The harmonic feature of the electric potential is preserved under the conformal maps, that is:

$$(\partial^2_{\tilde{u}} + \partial^2_{\tilde{v}})\tilde{\phi} = 0. \tag{27}$$

The original boundary conditions at the upper and lower shells are mapped to $\tilde{v} = 0$ and $\tilde{v} = \pi$, respectively. With these boundary conditions, and realizing that the potential needs to be constant along the u axis, the Laplace equation reduces to

$$\partial^2_{\tilde{v}}\tilde{\phi}(\tilde{v}) = 0 \implies \tilde{\phi}(\tilde{v}) = V_s\frac{\tilde{v}}{\pi}. \tag{28}$$

Finally, we revert from $\tilde{v}$ to α using Eq. (25), namely, $\tilde{v} = \alpha$, which implies

$$\phi(\alpha) = V_s \frac{\alpha}{\pi}. \tag{29}$$

Finally, note that α is the phase of the complex variable ω:

$$\omega = i\frac{1-z}{1+z} = i\frac{(1-z)(1+z^*)}{(1+z)(1+z^*)} = \frac{2y + i(1-x^2-y^2)}{(1+x)^2+y^2}, \tag{30}$$

which gives

$$\phi(\alpha) = \varphi(x,y) = \frac{V_s}{\pi}\arctan\left(\frac{1-x^2-y^2}{2y}\right). \tag{31}$$

[1] J. W. Brown and R. V. Churchill, *Complex variables and applications*. McGraw-Hill Higher Education, 2004 [Online]. Available: https://books.google.com/books?id=4vfuAAAAMAAJ