Dirac-Delta with a function inside

2024-07-01

Abstract

We will have some fun with $\delta\left(f(x)\right)$, and simplify it. Find the web-page here.

This will be a quick post on handling functions inside Dirac-delta function. Let us start simple and figure out the constant scaling of the argument first, i.e., $\delta(\alpha\, x)$. This will certainly be proportional to $\delta(x)$ since the zero is still at $x = 0$. But it will pick up an overall factor since Dirac delta function is defined as a density, i.e., it integrates to 1. Let's see what $\delta(\alpha\, x)$ does under the integral:

$$\lim_{R\to\infty}\int_{-R}^{R} dx\delta(\alpha\, x) = \lim_{R\to\infty}\int_{-R/\alpha}^{R/\alpha} d\left(\frac{y}{\alpha}\right)\delta(y) = \frac{1}{\alpha}\lim_{R\to\infty}\int_{-R/\alpha}^{R/\alpha} dy\delta(y) = \begin{cases} \frac{1}{\alpha}, & \text{if } \alpha > 0 \\ -\frac{1}{\alpha}, & \text{if } \alpha < 0 \end{cases} = \frac{1}{|\alpha|}, \tag{1}$$

where the negative sign appears since we have to flip the integral limits when $\alpha < 0$.

Now Consider a function $f(x)$ which has its zeros at points x_i. Now we just stick this inside $\delta()$, and it will have its peaks at $x = x_i$:

$$\delta\left(f(x)\right) \quad = \quad \sum_i c_i \delta(x - x_i), \tag{2}$$

and the goal is to find the c_i's. We expand $f(x)$ around x_i as $f(x) = (x - x_i)f'(x_i)$ when x is in the vicinity if x_i. This gives

$$\delta\left(f(x)\right)_{x\sim x_i} \quad = \quad \delta\left((x - x_i)f'(x_i)\right) = \frac{1}{|f'(x_i)|}\delta\left(x - x_i\right), \tag{3}$$

wher we used Eq. (1). Putting it all together, we get:

$$\delta\left(f(x)\right) \quad = \quad \sum_i \frac{1}{|f'(x_i)|}\delta\left(x - x_i\right), \tag{4}$$

which expresses the Dirac-delta of a function as a series of basic Dirac-delta functions.