

Musings on the Gamma Distribution

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Abstract

Gamma distribution appears as we add exponentially distributed random numbers. We show this using two different methods, including Laplace transforms. Find the web-page here.

$S_n = \sum_{i=1}^n T_i$ is a sum of n random numbers. It is illustrative to consider $n = 2$ case and figure out the distribution of the sum of two random numbers T_1 and T_2 . The cumulative probability density of $S_2 \equiv T_1 + T_2$ is given by:

$$\begin{aligned} F_{S_2}(t) &= P(T_1 + T_2 < t) = \int_{t_1+t_2 < t} f_{T_1}(t_1) f_{T_2}(t_2) dt_1 dt_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{t-t_2} f_{T_2}(t_2) dt_2 f_{T_1}(t_1) dt_1 \\ &= \int_{-\infty}^{\infty} F_{T_2}(t-t_1) f_{T_1}(t_1) dt_1. \end{aligned} \quad (1)$$

The probability density function is the derivative of Eq. (1):

$$f_{S_2}(t) = \frac{d}{dt} F_{S_2}(t) = \int_{-\infty}^{\infty} f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1 = \int_0^t f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1, \quad (2)$$

where the limits of the integral are truncated to the range where $f \neq 0$. The integral in Eq.(2) is known as the convolution integral:

$$f_{T_1} \otimes f_{T_2} \equiv \int_{-\infty}^{\infty} f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1. \quad (3)$$

In the special case of exponential distributions, f is parameterized by a single parameter λ , which represents the failure rate, and it is given by

$$f_T(t) = \lambda e^{-\lambda t}, \quad t > 0. \quad (4)$$

From Eq. (2) we get:

$$f_{S_2}(t) = \int_0^t f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1 = \lambda^2 \int_0^t e^{-\lambda(t-t_1)} e^{-\lambda t_1} dt_1 = \lambda^2 e^{-\lambda t} \int_0^t dt_1 = \lambda^2 t e^{-\lambda t}, \quad (5)$$

which is actually a Γ distribution. The corresponding cumulative failure function is:

$$\begin{aligned} F_{S_2}(t) &= \int_0^t d\tau f_{S_2}(\tau) = \lambda^2 \int_0^t d\tau \tau e^{-\lambda \tau} = -\lambda^2 \frac{d}{d\lambda} \left[\int_0^t d\tau e^{-\lambda \tau} \right] = \lambda^2 \frac{d}{d\lambda} \left[\frac{e^{-\lambda t} - 1}{\lambda} \right] \\ &= 1 - e^{-\lambda t} (1 + \lambda t). \end{aligned} \quad (6)$$

This is pretty neat. Can we move to the next level and add another T_i , i.e., $S_3 = T_1 + T_2 + T_3 = S_2 + T_3$. We just reiterate Eq. (2) with probability density for S_2 from Eq. (5).

$$f_{S_3}(t) = \int_0^t f_{T_3}(t-t_1) f_{S_2}(t_1) dt_1 = \lambda^3 \int_0^t e^{-\lambda(t-t_1)} t_1 e^{-\lambda t_1} dt_1 = \lambda^3 \frac{t^2}{2} e^{-\lambda t}, \quad (7)$$

which was very easy! In fact, we can keep adding more terms. The exponentials kindly drop out of the t_1 integral, and we will be simply integrating powers of t_1 , and for $S_n \equiv T_1 + T_n + \dots + T_n$ to get:

$$f_{S_n}(t) = \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t}. \quad (8)$$

It will be fun if we redo this with some advanced mathematical tools, such as the Laplace transform, which is defined as:

$$\tilde{f}(s) \equiv \mathcal{L}[f(t)] = \int_0^\infty dt e^{-st} f(t). \quad (9)$$

There are a couple of nice features of the Laplace transforms we can make use of. The first one is the mapping of convolution integrals in t space to multiplication in s space. To show this, let's take the Laplace transform of Eq. (3):

$$\mathcal{L}[f_{T_1} \otimes f_{T_2}] = \int_0^\infty dt e^{-st} \int_{-\infty}^\infty f_{T_2}(t-t_1) f_{T_1}(t_1) dt_1 = \int_{-\infty}^\infty dt_1 \int_0^\infty dt e^{-s(t-t_1)} f_{T_2}(t-t_1) e^{-st_1} f_{T_1}(t_1). \quad (10)$$

Let's take a closer look at the middle integral:

$$\int_0^\infty dt e^{-s(t-t_1)} f_{T_2}(t-t_1) = \int_{-t_1}^\infty dt e^{-s\tau} f_{T_2}(\tau) = \int_0^\infty d\tau e^{-s\tau} f_{T_2}(\tau) = \tilde{f}_{T_2}(s), \quad (11)$$

where we first defined $\tau = t - t_1$, and then shifted the lower limit of the integral back to 0 since $f_{T_2}(t) = 0$ for $t < 0$. Putting this back in, we have the nice property:

$$\mathcal{L}[f_{T_1} \otimes f_{T_2}] = \tilde{f}_{T_1}(s) \tilde{f}_{T_2}(s). \quad (12)$$

How do we make use of this? The probability distribution of a sum of random numbers is the convolution of individual distributions:

$$f_{S_n} = \underbrace{f_{T_1} \otimes f_{T_2} \otimes \dots \otimes f_{T_n}}_{n \text{ times}}. \quad (13)$$

We can map this convolution to multiplications in s space:

$$\tilde{f}_{S_n}(s) \equiv \mathcal{L}[f_{S_n}] = \underbrace{\tilde{f}_{T_1} \tilde{f}_{T_2} \dots \tilde{f}_{T_n}}_{n \text{ times}} = \prod_{j=1}^n \tilde{f}_{T_j}. \quad (14)$$

When the individual random numbers are independent and have the same distribution, we get:

$$\tilde{f}_{S_n}(s) = (\tilde{f}_{T_j})^n. \quad (15)$$

If the random numbers are exponentially distributed, as in Eq. (4), their Laplace transformation is easy to compute:

$$\tilde{f}(s) = \int_0^\infty dt e^{-st} \lambda e^{-\lambda t} = \frac{\lambda}{s + \lambda}, \quad (16)$$

which mean the Laplace transform of the sum is:

$$\tilde{f}_{S_n}(s) = \left(\frac{\lambda}{s + \lambda} \right)^n. \quad (17)$$

We will have to inverse transform Eq. (17), which will require some trick. This brings us to the second nifty property of Laplace transform. Consider transforming $tf(t)$:

$$\mathcal{L}[tf(t)] = \int_0^\infty dt te^{-st} f(t) = -\frac{d}{ds} \left[\int_0^\infty dt e^{-st} f(t) \right] = -\frac{d}{ds} [\tilde{f}(s)]. \quad (18)$$

Therefore, we see that Laplace transform maps the operation of multiplying with t to taking negative derivatives in s space:

$$t \iff -\frac{d}{ds}. \quad (19)$$

We re-write Eq. (17) as:

$$\tilde{f}_{S_n}(s) = \left(\frac{\lambda}{s + \lambda} \right)^n = \frac{\lambda^n}{(n-1)!} \left(-\frac{d}{ds} \right)^n \left(\frac{\lambda}{s + \lambda} \right). \quad (20)$$

Using the property in Eq. (19), we can invert the transform:

$$f_{S_n}(t) = \mathcal{L}^{-1}[f_{S_n}] = \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad (21)$$

which is what we got earlier in Eq. (8).