Radial Green's Function in Cylindrical Coordinates

2023-12-04

Abstract

Deriving the Green's function for the radial part in cylindrical coordinates.

Contents

The Laplace operator in cylindrical coordinates reads

$$\vec{\nabla}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{1}$$

The Green's function G is defined as the solution of the following differential equation:

$$\vec{\nabla}^2 G(\vec{r}) = \delta^3(\vec{r}) = \frac{1}{r}\delta(r)\delta(\theta)\delta(z). \tag{2}$$

We will consider problems with no θ and z dependence, and that will leave behind the r dependence. Let's solve this for $r > 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0 \implies r\frac{\partial G}{\partial r} = C \implies G(r) = C\ln r + D, \tag{3}$$

where C and D are constants. The term D is not that interesting since it will drop up on being acted on by $\frac{d}{dr}$, and therefore we can set it to 0. However, we do need to figure out what C is. Although, $\ln(r)$ diverges at $r = 0$, the overall expression with the derivatives vanishes. This will require a lot of care to handle the singularity properly. To this end, let's protect the function by introducing a parameter ϵ, which we will set to zero when all is said and done:

$$G_\epsilon(r) \equiv C\ln(r+\epsilon), \tag{4}$$

Inserting this back into Eq. (3), we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_\epsilon}{\partial r}\right) = \frac{C}{r}\frac{\partial}{\partial r}\left(\frac{r}{r+\epsilon}\right) = \frac{C}{r}\frac{\partial}{\partial r}\left(1 - \frac{\epsilon}{r+\epsilon}\right) = \frac{\epsilon C}{r(r+\epsilon)^2} = \frac{1}{r}\delta(r)\delta(\theta)\delta(z). \tag{5}$$

Integrating this over whole space:

$$\int_0^\infty dr\int_0^{2\pi} d\theta\int_{-h/2}^{h/2} dz\, r\frac{\epsilon C}{r(r+\epsilon)^2} = 2\pi hC = \int_0^\infty dr\int_0^{2\pi} d\theta\int_{-h/2}^{h/2} dz\frac{r}{r}\delta(r)\delta(\theta)\delta(z) = 1 \implies C = \frac{1}{2\pi h}, \tag{6}$$

where h is an arbitrary length along the z axis. Note that we need h so that the units make sense. In pure mathematical expression, one wouldn't care about it. However, imagine that r has the unit of length L. ∇^2 has the unit of L^{-2}. $\delta^3(\vec{r})$ has the unit of L^{-3}. To match the units, G has the have the unit as L^{-1}, which comes from the coefficient C. h can be set to 1, a unitless value, and the unit can be absorbed into G to get:

$$G(r) = \frac{1}{2\pi}\ln(r). \tag{7}$$

The ϵ business is a tricky one, and we could have avoided it if we invoked the Gauss theorem by integrating Eq. (2):

$$\int_{\mathcal{V}} dV \vec{\nabla} \cdot \left(\vec{\nabla} G(\vec{r}) \right) = \int_{\partial \mathcal{V}} d\vec{S} \cdot \vec{\nabla} G(\vec{r}) = \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} dz r \frac{d}{dr} (C ln(r)) = 2\pi h C = \int_{\mathcal{V}} dV \delta^3(\vec{r}) = 1, \tag{8}$$

which would return the same value of C.