

Integral of the month: $\oint_c \frac{|dz|}{|z-\beta|^2}$

Abstract

Going around a circle!

Index Terms

Integral, Residue Calculus, Branch Cut

We want to compute the integral $I = \oint_c \frac{|dz|}{|z-\beta|^2}$ where $|\beta| < 1$ and the contour c is a unit circle centered at the origin. We will show below that the integrand has two poles: one inside the contour and another outside, as shown in Fig. 1.

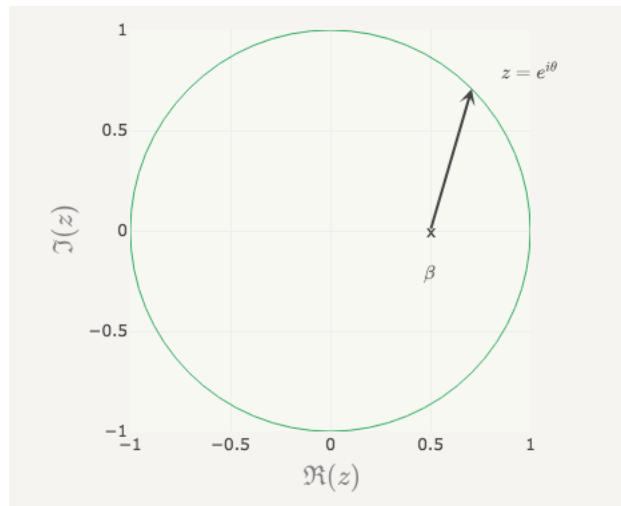


Figure 1: The location of the roots for $\beta = 1/2$. This is a static copy, find the interactive HTML-document [here](#).

I. DEALING WITH ABSOLUTE VALUE OPERATIONS

The pesky thing about this integral is the existence of absolute value operations which typically result in square roots of sum of squares. That is a pain in the neck. However, since the contour is just a lovely circle, we can avoid all that. This is because $z = e^{i\theta}$ and $z^* = e^{-i\theta} = z^{-1}$. Similarly $|dz| = |i d\theta e^{i\theta}| = |d\theta| = d\theta$. Furthermore, since $dz = i d\theta e^{i\theta} = iz d\theta$, we get $|dz| = d\theta = \frac{dz}{iz}$. Putting it all together, we can get rid of the absolute values

$$I = \oint_c \frac{dz}{iz(z-\beta)(z^*-\beta^*)} = \oint_c \frac{dz}{iz(z-\beta)(z^{-1}-\beta^*)} = \oint_c \frac{dz}{i(z-\beta)(1-\beta^*z)}. \quad (1)$$

Note that if $\beta = 0$, the original integral reduces to $\oint_c d\theta$ or equivalently $\oint_c \frac{dz}{iz}$, both of which give 2π . Once we have the final expression for the result, we should get 2π as $\beta \rightarrow 0$. Before we move on, note that the only constraint on β is $|\beta| < 1$. It is not necessarily a real number. However, note the rotational invariance of the integral: if β has an imaginary component, $\beta = e^{i\alpha}|\beta|$, we can redefine $z = \tilde{z}e^{i\alpha}$ and all the phases will disappear due to the absolute values and the invariance of the circle under phase rotations. Therefore, without any loss of generality we could assume β is a real number, or patiently follow through the math to see that β will appear as $|\beta|$ in the final answer.

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Find the interactive HTML-document [here](#).

The integrand has poles at $z = \beta$ and $z = 1/\beta^*$. As $|\beta| < 1$, $1/\beta^*$ will be outside of the circle. Then it is easy to evaluate the integral:

$$I = 2\pi i \text{Residue}(f(z = \beta)) = \frac{2\pi}{1 - |\beta|^2}. \quad (2)$$

II. AN ALTERNATIVE FORM

In most cases such integrals are given as angle integral with integrands having sin or cos terms in the denominator. Let's start from the original expression and rewrite it as an angle integral with the change of variable $z = e^{i\theta}$, which gives $|z - \beta|^2 = (e^{i\theta} - \beta)(e^{-i\theta} - \beta) = 1 - 2\beta \cos \theta + \beta^2$. The integral becomes

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2\beta \cos \theta + \beta^2}, \quad (3)$$

and that is the form many problems start from. To solve such a problem, one would just undo the transformation $e^{i\theta} \equiv z$, and $\cos \theta = \frac{z+z^{-1}}{2}$ and $\int_0^{2\pi} d\theta \rightarrow \oint_c \frac{dz}{iz}$.