Integral of the month: $\int dr \cos r^2$

The Fresnel integrals are defined as follows:

$$\begin{aligned} S(t) &= \int_0^t dr \sin r^2, \\ C(t) &= \int_0^t dr \cos r^2. \end{aligned} \quad (1)$$

For a general value of t, the integrals need to be evaluated numerically. However, the asymptotic values $C(t)$ and $S(t)$ can be calculated via the closed contour integral below:

$$I = \oint_C dz e^{-z^2}, \quad (2)$$

where the contour C is illustrated in Fig. 1.

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Find the interactive HTML-document here.

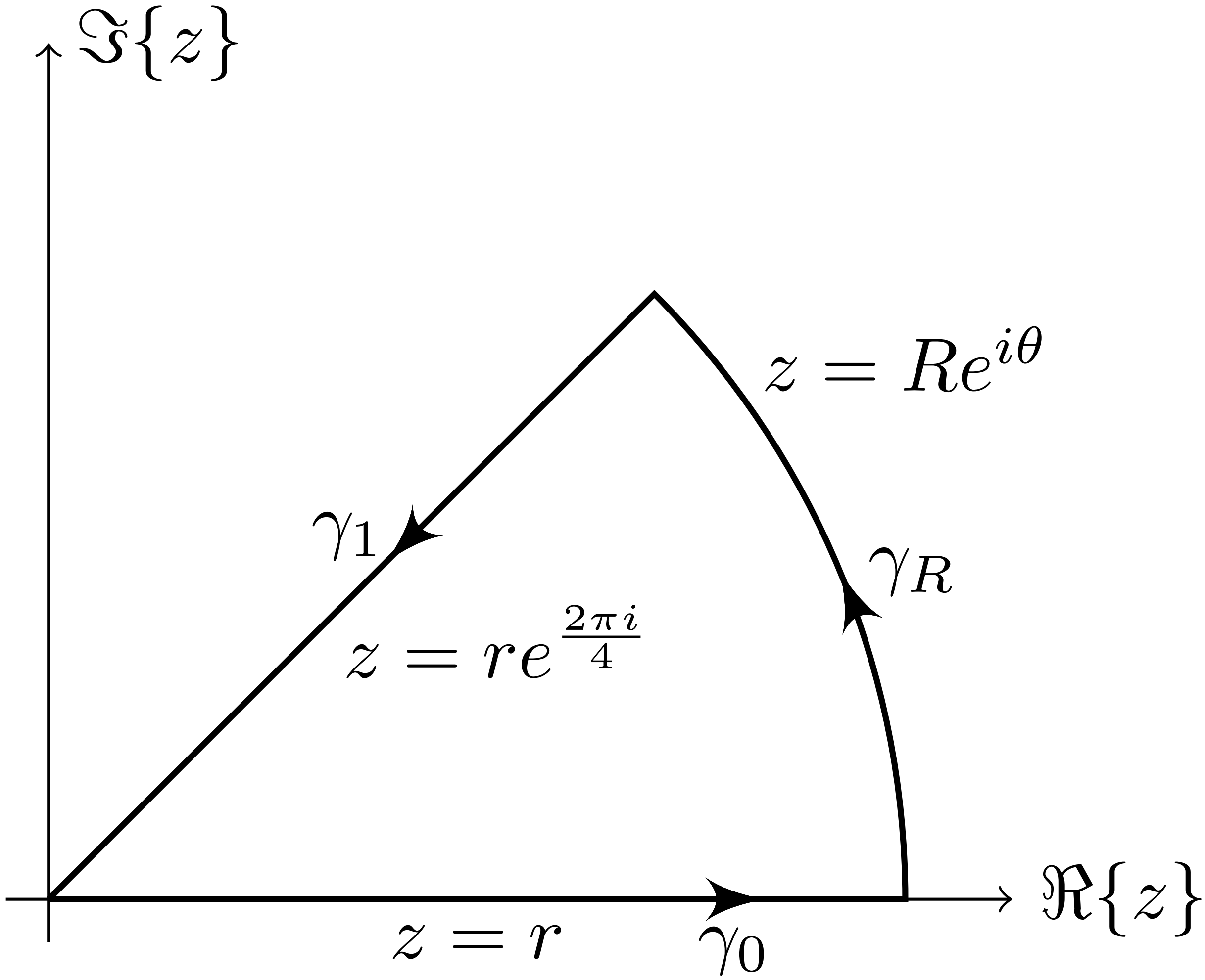

Fig. 1: The contour to evaluate the integral in Eq.
eqrefeq:cint. The return path, γ_1, is chosen such that the integrand reduces to the regular Gaussian.

Let's first evaluate the integral on γ_0 in the limit $R \to \infty$:

$$I_{\gamma_0} = \lim_{R\to\infty} \int_0^R dr e^{-r^2} = \frac{\sqrt{\pi}}{2}, \tag{3}$$

where the details of the derivation can be found here. Now consider the (absolute value of the) integral on γ_R in the limit $R \to \infty$:

$$\begin{aligned} |I_{\gamma_R}| &= \left| \lim_{R\to\infty} R \int_0^{\frac{\pi}{4}} d\theta e^{i\theta} e^{-R^2(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta)} \right| = \left| \lim_{R\to\infty} R \int_0^{\frac{\pi}{4}} d\theta e^{i\theta + i\sin(2\theta)} e^{-R^2\cos(2\theta)} \right| \\ &\le \left| \lim_{R\to\infty} R \int_0^{\frac{\pi}{4}} d\theta e^{-R^2\cos(2\theta)} \right|. \end{aligned} \tag{4}$$

Let's try to put a bound on $\cos(2\theta)$ in the range $0 \le \theta \le \pi/4$. At $\cos(2\theta)|_{\theta=0} = 1$ and $\cos(2\theta)|_{\theta=\pi/4} = 0$. We can draw a line that connects these two points: $1 - \frac{4\theta}{\pi}$. Since $\frac{d^2}{d\theta^2}\cos(2\theta) = -4\cos(2\theta) < 0$ for $0 < \theta < \pi/4$, we know that $\cos(2\theta) < 1 - \frac{4\theta}{\pi}$ in this range. This observation is illustrated in Fig. 2.

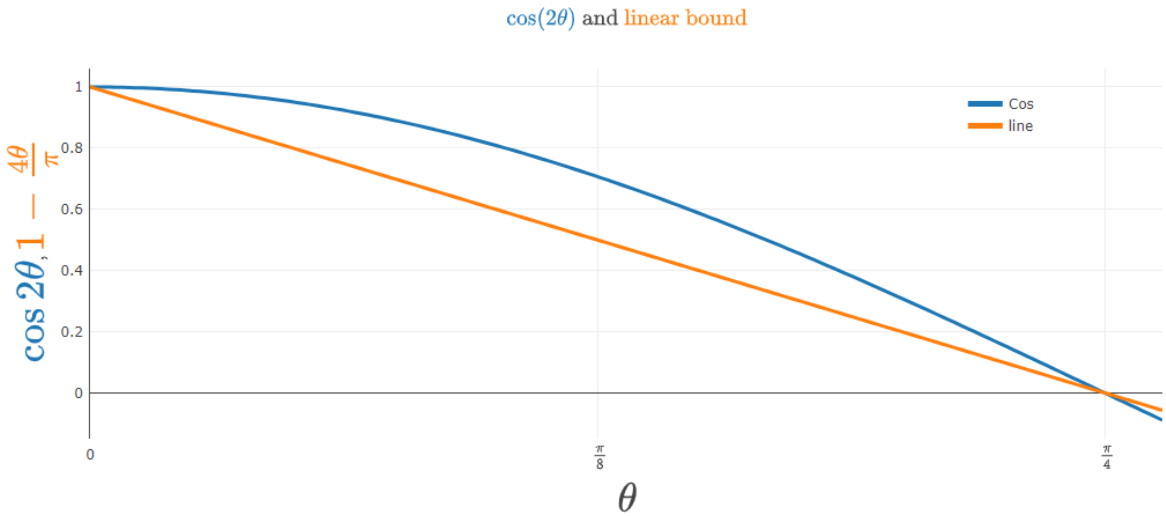

Fig. 2: $\cos(2\theta)$ and a bound on it with the line $1-\frac{4\theta}{\pi}$.

We can now go back to Eq. (6) make use of the bound:

$$\begin{aligned} |I_{\gamma_R}| &\leq \left|\lim_{R\to\infty} R\int_0^{\frac{\pi}{4}} d\theta e^{-R^2\cos(2\theta)}\right| \leq \left|\lim_{R\to\infty} R\int_0^{\frac{\pi}{4}} d\theta e^{-R^2\left(1-\frac{4\theta}{\pi}\right)}\right| = \left|\lim_{R\to\infty} Re^{-R^2}\int_0^{\frac{\pi}{4}} d\theta e^{R^2\frac{4\theta}{\pi}}\right| \\ &\leq \left|\lim_{R\to\infty} R\frac{\pi}{4R^2}\left(1-e^{-R^2}\right)\right| = \left|\lim_{R\to\infty}\frac{\pi}{4R}\left(1-e^{-R^2}\right)\right| = 0. \end{aligned} \tag{5}$$

Finally, let's look at the integral on γ_1 in the limit $R\to\infty$:

$$\begin{aligned} I_{\gamma_1} &= \lim_{R\to\infty} R\int_R^0 dre^{\frac{i\pi}{4}}e^{-r^2\frac{i\pi}{2}} = -\frac{1+i}{\sqrt{2}}\lim_{R\to\infty}\int_0^R dr\left(\cos r^2 - i\sin r^2\right) = -\frac{1+i}{\sqrt{2}}\int_0^\infty dr\left(\cos r^2 - i\sin r^2\right) \\ &= -\frac{1}{\sqrt{2}}\left(\int_0^\infty dr\cos r^2 + \int_0^\infty dr\sin r^2 + i\left[\int_0^\infty dr\cos r^2 - \int_0^\infty dr\sin r^2\right]\right). \end{aligned} \tag{6}$$

As we have computed individual pieces of the integral Eq.(2), we can assemble them and state that they need to add to 0 since e^{-z^2} is analytic everywhere. Therefore we have:

$$\begin{aligned} \oint_C dze^{-z^2} = 0 &= I_{\gamma_0} + I_{\gamma_R} + I_{\gamma_1} \\ &= \frac{\sqrt{\pi}}{2} + 0 - \frac{1}{\sqrt{2}}\left(\int_0^\infty dr\cos r^2 + \int_0^\infty dr\sin r^2 + i\left[\int_0^\infty dr\cos r^2 - \int_0^\infty dr\sin r^2\right]\right). \end{aligned} \tag{7}$$

Matching the real and imaginary parts, we get:

$$\begin{aligned} \frac{1}{\sqrt{2}}\left(\int_0^\infty dr\cos r^2 + \int_0^\infty dr\sin r^2\right) &= \frac{\sqrt{\pi}}{2}, \\ \frac{1}{\sqrt{2}}\left(\int_0^\infty dr\cos r^2 - \int_0^\infty dr\sin r^2\right) &= 0, \end{aligned} \tag{8}$$

from which we get

$$\int_0^\infty dr\cos r^2 = \int_0^\infty dr\sin r^2 = \frac{1}{2}\sqrt{\frac{\pi}{2}} \simeq 0.626. \tag{9}$$

Now we can conclude with the plots of the Fresnel integrals in Fig. 3.

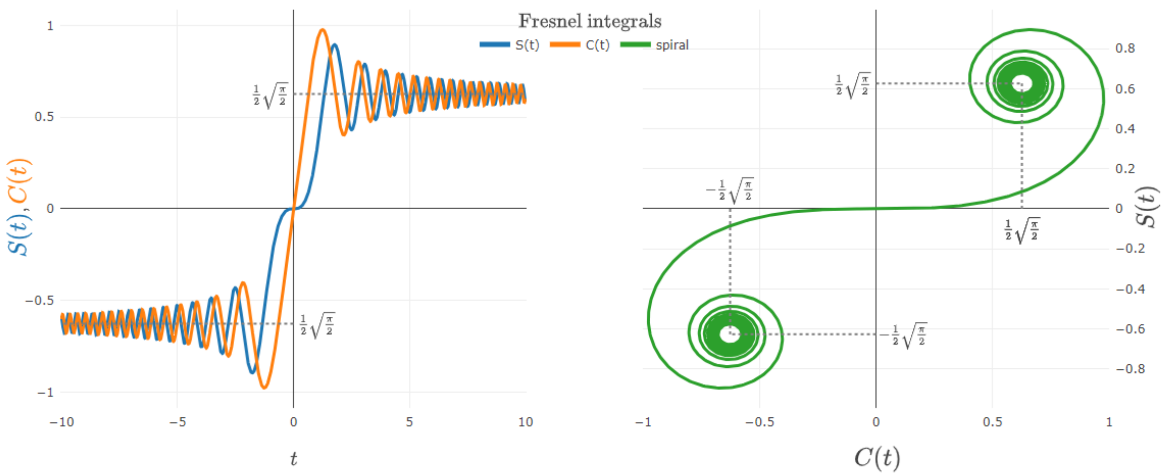

Fig. 3: Left: Fresnel integrals as a function of their argument, Right: parametric plot of the integrals forming the Euler spiral.