Integral of the month: $\iint_S dt'dtf(t'-t)$

Abstract

When the function depends only on the difference of the parameters...

Index Terms

Integral, Wiener-Khinchin theorem

We want to compute the integral $I = \int_{\frac{-T}{2}}^{\frac{T}{2}}\int_{\frac{-T}{2}}^{\frac{T}{2}} dt'dtf(t'-t)$, which appears frequently in Fouirer transforms. For instance, the proof of Wiener-Khinchin theorem requires evaluation of such an integral.

The argument of f begs for a change of coordinates:

$$u = t' - t, \quad \text{and} \quad v = t + t', \tag{1}$$

and the associated inverse transform reads:

$$t' = \frac{u+v}{2}, \quad \text{and} \quad t' = \frac{v-u}{2}. \tag{2}$$

This transformation will rotate and scale the integration domain as shown in Fig. 1.

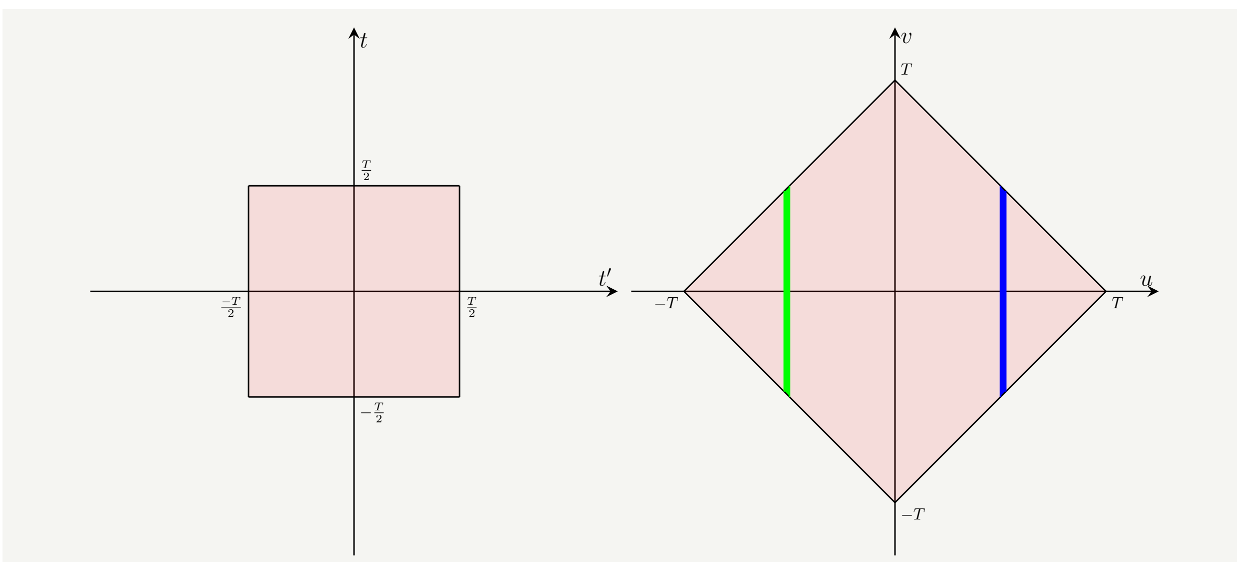

Figure 1: The integration domain in the $t-t'$ domain (left) and $u-v$ domain(right). Since there is no v dependence, v integration gives the height of the green and blue slices.

The equation of the top boundary on the right can be written as $v = T - u$, and on the left as $ v= T+u$. We can actually combine them as $v = T - |u|$. We can do the same analysis for the lower boundaries to see that the height of the slices at a given u is $2(T - |u|)$. This will help us easily integrate v out as follows:

$$\begin{aligned} I &= \int_{\frac{-T}{2}}^{\frac{T}{2}}\int_{\frac{-T}{2}}^{\frac{T}{2}} dt'dtf(t'-t) = \iint_{S_{u,v}} \left|\frac{\partial(t,t')}{\partial(u,v)}\right| dvduf(u) \\ &= \int_{-T}^{T} 2(T-|u|)\times\frac{1}{2}dvduf(u) = \int_{-T}^{T} duf(u)(T-|u|), \end{aligned} \tag{3}$$

email: quarktetra@gmail.com
Find the interactive HTML-document here.

where $\left|\frac{\partial(t,t')}{\partial(u,v)}\right| = \frac{1}{2}$ is the determinant of the Jacobian matrix associated with the transformation in Eq. (2).