Integral of the month: $\int dx \frac{\sin x}{x}$

Abstract

Three different ways of evaluating this lovely integral!

Index Terms

Integral, Residue Calculus, Branch Cut

We want to compute the integral $I = \int_{-\infty}^{\infty} dx \frac{\sin x}{x}$ in various ways.

I. A complex contour integration

As it is typically done, we first upgrade real valued parameter x to a complex number z and then construct the contour in Fig. 1.

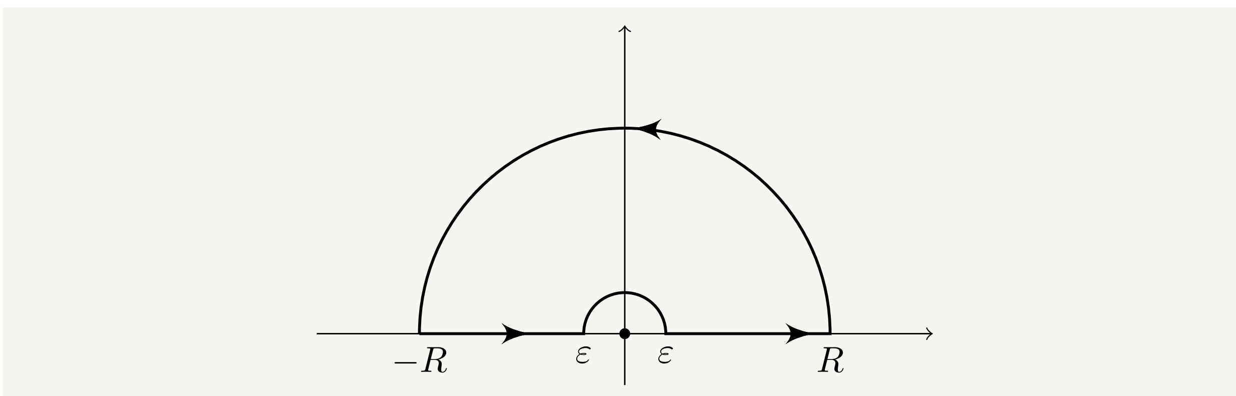

Figure 1: The complex contour in which the singularity at the origin is avoided by bending the curve around it.

On the circle of radius ε, $z = \varepsilon e^{i\theta}$ where $\theta \in [0, \pi]$. And on the large circle $z = Re^{i\phi}$ where $\phi \in [0, \pi]$. We can easily evaluate the following integral (in the limit $\varepsilon \to 0$ and $R \to \infty$):

$$\begin{aligned} I_c &\equiv \oint dz \frac{e^{iz}}{z} = \int_{-R}^{\varepsilon} dx \frac{e^{ix}}{x} + \int_{\pi}^{0} \varepsilon i d\theta e^{i\theta} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} + \int_{\varepsilon}^{R} dx \frac{e^{ix}}{x} + \int_{0}^{\pi} R i d\phi e^{i\phi} \frac{e^{iRe^{i\phi}}}{Re^{i\phi}} \\ &= \int_{-R}^{R} dx \frac{e^{ix}}{x} + \int_{\pi}^{0} i d\theta + \int_{0}^{\pi} i d\phi e^{iRe^{i\phi}} = \int_{-R}^{R} dx \frac{e^{ix}}{x} - i\pi. \end{aligned} \tag{1}$$

Note that the integral over the large circle vanishes as $R \to \infty$ since $e^{iRe^{i\phi}} = e^{-R\sin\phi} e^{iR\cos\phi}$. Therefore, by explicit evaluation, we see that $I_c = \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} + i\pi$. But, from the theory of residues, we know that the closed loop integral of a function is 0 if the contour does not enclose any poles. Therefore

$$\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi, \tag{2}$$

and if we take the imaginary parts of the bothsides, we get

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi. \tag{3}$$

Side note: we evaluated the integral over the inner half circle explicitly. We could also see that it would give $i\pi$ by observing that it is half of a circle that would have enclosed the singularity at the origin. Integral over the full circle would give $2\pi i$, and the integral over the upper-half gives $i\pi$.

email: quarktetra@gmail.com
Find the interactive HTML-document here.

II. Parametric Laplace transform

One of my favorite tricks in integration is to introduce a parameter in the integrand and manipulate it to simplify the integral. Let us insert an α parameter in sin:

$$I(\alpha) = \int_{-\infty}^{\infty} dx \frac{\sin(\alpha x)}{x}. \tag{4}$$

Let us apply a Laplace transform with respect to α to be followed by the inverse Laplace transform

$$\begin{aligned} I &= \mathscr{L}^{-1}\left[\mathscr{L}[I(\alpha)]\right] = \mathscr{L}^{-1}\left[\int_{-\infty}^{\infty} dx \mathscr{L}\left[\frac{\sin(\alpha x)}{x}\right]\right] = \mathscr{L}^{-1}\left[\int_{-\infty}^{\infty} dx \frac{1}{s^2+x^2}\right] \\ &= \mathscr{L}^{-1}\left[\frac{1}{s}\int_{-\infty}^{\infty} d(x/s)\frac{1}{1+(x/s)^2}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}\arctan(x/s)\Big|_{-\infty}^{\infty}\right] = \pi\mathscr{L}^{-1}\left[\frac{1}{s}\right] \\ &= \pi. \end{aligned} \tag{5}$$

III. Direct Laplace transform

Here is a reminder on the definition of the Laplace transform:

$$F(s) = \mathscr{L}[f] = \int_0^{\infty} dx e^{-sx} f(x). \tag{6}$$

From the definition, we can see that we can create a $\frac{1}{x}$ term in the integrand if we simply integrate left side from s to ∞:

$$\int_s^{\infty} d\tilde{s} F(\tilde{s}) = \int_0^{\infty} dx \left[\int_s^{\infty} d\tilde{s} e^{-\tilde{s}x}\right] f(x). = \int_0^{\infty} dx e^{-sx}\frac{f(x)}{x}. \tag{7}$$

Therefore, if we have an expression of the form $f(x)/x$, we can transform it as $\int_s^{\infty} d\tilde{s} F(\tilde{s})$. In our case $f(x) = \sin x$ and $F(s) = \frac{1}{1+s^2}$. Using the property above we get

$$I_s = \int_0^{\infty} dx e^{-sx}\frac{\sin x}{x} = \int_s^{\infty} d\tilde{s} F(\tilde{s}) = \int_s^{\infty} d\tilde{s}\frac{1}{1+\tilde{s}^2} = \arctan\tilde{s}\Big|_s^{\infty} = \pi/2 - \arctan s. \tag{8}$$

Note that I_s at $s = 0$ is half of the integral we are looking for: $\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = 2\int_0^{\infty} dx \frac{\sin x}{x}$. Doubling the result at $s = 0$ yields:

$$I = 2I_0 = \pi - 2\arctan(0) = \pi \tag{9}$$

There you have it, three ways of evaluating this lovely integral.