One-dimensional attractive potential of any shape supports a bound state

Abstract

We use the variational principle to prove that one-dimensional attractive potential of any shape supports a bound state.

Index Terms

variational principle, bound states

Given that a one-dimensional attractive finite square well has at least one bound state, we can use the variational principle to prove that there is at least one bound state for a one-dimensional attractive potential of any shape. Assume that the potential $V(x)$ is negative definite. One can fit a square well into this potential. We choose the well such that the potential lies under the well, i.e, it is deeper than the well. Let's define the difference in between by $\tilde{V}(x) = V(x) - V_W(x)$ where $V_W(x)$ is the well potential, and it is important to note that $\tilde{V}(x) \leq 0$ Now, using the variational principle we can find an upper limit for the ground state of the Hamiltonian $H = p^2/2m + V(x)$. Let's calculate the expectation value of H in the bound state of the well problem. Defining this state as $|\psi_W\rangle$, we have

$$\langle H \rangle_W = \langle \psi_W | H | \psi_W \rangle = \langle \psi_W | p^2/2m + V_W(x) | \psi_W \rangle + \langle \psi_W | \tilde{V}_W(x) | \psi_W \rangle = E_{0W} + \langle \tilde{V}_W(x) \rangle. \tag{1}$$

Both terms in the final expression are negative which makes $\langle H \rangle_W < 0$. On top of this information, the variational principle tells us that the true ground state energy of H is lower than or equal to any values you get with trial wave functions. Therefore, we conclude that the ground state energy of H is negative, which entails that it supports bound states.

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