Coherent states of Quantum Harmonic Oscillator

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Abstract

A derivation of the coherent states of quantum harmonic oscillator.

Contents

1 Introduction

This post builds on an earlier post of mine, Quantum Harmonic Oscillator. The two main references I used are [1] and [2].

Assume that there exists a state $|\alpha\rangle$ which is an eigenstate of the lowering operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

where α is a complex number we will need to calculate.

We can quickly compute the expected values of $\hat{x}$ and $\hat{p}$:

$$\langle\hat{x}\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}\langle a + a^\dagger\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*) = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\Re(\alpha), \tag{2}$$

and

$$\langle\hat{p}\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}\langle a - a^\dagger\rangle_\alpha = -i\sqrt{\frac{m\hbar\omega}{2}}(\alpha - \alpha^*) = \sqrt{m\hbar\omega}\sqrt{2}\Im(\alpha). \tag{3}$$

And the expected values of $\hat{x}^2$ and $\hat{p}^2$ are:

$$\langle\hat{x}^2\rangle_\alpha = \frac{\hbar}{2m\omega}\langle(a + a^\dagger)^2\rangle_\alpha = \frac{\hbar}{2m\omega}[(\alpha + \alpha^*)^2 + 1], \tag{4}$$

and

$$\langle\hat{p}^2\rangle_\alpha = -\frac{m\hbar\omega}{2}\langle(a - a^\dagger)^2\rangle_\alpha = \frac{m\hbar\omega}{2}[1 - (\alpha - \alpha^*)^2] \tag{5}$$

The uncertainity relations become

$$\begin{aligned} \langle(\Delta x)^2\rangle_\alpha &= \langle\hat{x}^2\rangle_\alpha - (\langle\hat{x}\rangle_\alpha)^2 = \frac{\hbar}{2m\omega}, \\ \langle(\Delta p)^2\rangle_n &= \langle\hat{p}^2\rangle_\alpha - (\langle\hat{p}\rangle_\alpha)^2 = \frac{\hbar m\omega}{2}, \\ \Delta x\Delta p &= \frac{\hbar}{2}. \end{aligned} \tag{6}$$

$\frac{\hbar}{2}$ is equal to uncertainty value for the ground state.

Since the energy eigenstates, $|n\rangle$, form a complete basis, we should be able to express any state as a combination of $|n\rangle$'s as follows:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle. \tag{7}$$

We can isolate the coefficients by projecting the sum onto state $\langle m|$ and use the orthoganality of the basis vectors to collapse the summation:

$$\langle m|\alpha\rangle = \sum_{n=0}^{\infty} c_n \langle m|n\rangle = \sum_{n=0}^{\infty} c_n \delta_{m,n} = c_m \implies c_m = \langle m|\alpha\rangle. \tag{8}$$

Furthermore, we can express $\langle m|$ as $\left(\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle\right)^\dagger = \langle 0|\frac{a^m}{\sqrt{m!}}$ Inserting this back into Eq. (7), we get

$$|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{n!}}\langle 0|a^n|\alpha\rangle\right)|n\rangle = \sum_{n=0}^{\infty}\left(\langle 0|\alpha\rangle \frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle. \tag{9}$$

For $|\alpha\rangle$ to be normalized we will need the sum of the squares of the terms in the parenthesis to be unity:

$$\sum_{n=0}^{\infty} |\langle 0|\alpha\rangle|^2 \frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2} = 1 \implies \langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}. \tag{10}$$

Hence the final form of the coherent state $|\alpha\rangle$ is

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{11}$$

$|\alpha\rangle$ is a superposition of energy eigenstates and we can compute the expected value of n by using the number operator N:

$$\begin{aligned}\lambda \;\;&\equiv\;\; \langle\alpha|N|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{n!}\sqrt{m!}}\langle m|a^\dagger a|n\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m\alpha^n}{\sqrt{n!}\sqrt{m!}} n\delta_{m,n} \\ &= \;\; e^{-|\alpha|^2}\sum_{n=0}^{\infty} n\frac{|\alpha|^{2n}}{n!} = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\left[\beta\frac{d}{d\beta}\frac{\beta^n}{n!}\right]_{\beta=\alpha^2} = e^{-|\alpha|^2}\left[\beta\frac{d}{d\beta}e^{\beta}\right]_{\beta=|\alpha|^2} = |\alpha|^2. \end{aligned} \tag{12}$$

We can use the expected value of n, which we defined as λ to prescribe the probability distribution of n:

$$P(n) = \frac{\lambda^n}{n!}e^{-\lambda}, \tag{13}$$

which is nothing but the density function of the Poisson distribution. Let's check for the orthogonality of the coherent states

$$\langle\alpha|\alpha'\rangle \;\;=\;\; e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|\alpha'|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*m}}{\sqrt{m!}}\frac{\alpha'^n}{\sqrt{n!}}\underbrace{\langle m|n\rangle}_{\delta_{m,n}} = e^{(-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\alpha'|^2)}e^{\alpha^*\alpha'} = e^{-\frac{1}{2}|\alpha-\alpha'|^2}, \tag{14}$$

which shows that coherent states are not orthogonal. Alhough they are not mutually orthogonal, the set of coherent states are complete. Keep in mind that α is a complex number and it can be represented as $re^{i\theta}$. We can sweep through all the coherent states by sweeping through the parameters r and θ. Consider the following operator:

$$\begin{aligned}\int_0^\infty drr\int_0^{2\pi} d\theta|\alpha\rangle\langle\alpha| &= \sum_{m=0}^\infty\sum_{n=0}^\infty \frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}|n\rangle\langle m|\int_0^\infty drre^{-r^2}r^{m+n}\underbrace{\int_0^{2\pi}d\theta e^{i\theta(n-m)}}_{2\pi\delta_{m,n}} \\ &= \sum_{n=0}^\infty \frac{\pi}{n!}|n\rangle\langle n|\underbrace{\int_0^\infty d(r^2)re^{-r^2}(r^2)^n}_{n!} = \pi\sum_{n=0}^\infty |n\rangle\langle n| = \pi I. \end{aligned} \tag{15}$$

This shows that with proper normalization, we can create a unity operator using the coherent states:

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = I, \tag{16}$$

where $d^2\alpha$ stands for $rdrd\theta$.

We can further compactify the representation of the coherent state given in Eq. (11) by exponentiating the raising operator:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}}\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle. \tag{17}$$

We can project $|\alpha\rangle$ onto $|x\rangle$ to find how the coherent state looks in the coordinate representation:

$$\langle x|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle x|e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}\langle x|0\rangle, \tag{18}$$

where

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}. \tag{19}$$

The exponentiated operator is rather nontrivial to compute. Let's address that in a generic setting. Consider the following expression with two operators A and B:

$$\begin{aligned} e^{A+B} &= 1 + A + B + \frac{1}{2}\left(A^2 + B^2 + AB + BA\right) + \cdots \\ &= \left(1 + A + \frac{A^2}{2} + \cdots\right)\left(1 + B + \frac{B^2}{2} + \cdots\right)\left(1 - \frac{1}{2}[A,B] + \cdots\right) \\ &= e^A e^B e^{-\frac{1}{2}[A,B]}. \end{aligned} \tag{20}$$

In our current problem, we have $A = \alpha\sqrt{\frac{m\omega}{2\hbar}}x$ and $B = -\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}$, which gives $[A,B] = \frac{\alpha^2}{2}$.

$$\langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{1}{m\omega}\frac{d}{dx}\right)}e^{-\frac{m\omega x^2}{2\hbar}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{\alpha^2}{4}}e^{\alpha\sqrt{\frac{m\omega}{2\hbar}}x}e^{-\alpha\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}}e^{-\frac{m\omega x^2}{2\hbar}}. \tag{21}$$

Let's deal with the exponentiated derivative in a generic sense as follows:

$$e^{\kappa\frac{d}{dx}}f(x) = \sum_{n=0}^\infty \frac{\kappa^n}{n!}\frac{d^n}{dx^n}f(x) = f(x+\kappa), \tag{22}$$

where we observed that the series nothing but the Taylor expansion of $f(x+\kappa)$. This is no surprise since

$e^{\kappa\frac{d}{dx}}$ is nothing but the translation operator! Putting this back in we get:

$$\begin{aligned}
\langle x|\alpha\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}|\alpha|^2 - \frac{\alpha^2}{4} + \alpha\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega\left(x-\alpha\sqrt{\frac{\hbar}{2m\omega}}\right)^2}{2\hbar}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}|\alpha|^2 - \frac{\alpha^2}{2} + \sqrt{2}\alpha\sqrt{\frac{m\omega}{\hbar}}x - \frac{m\omega}{2\hbar}x^2\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{1}{2}\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{2}\Re(\alpha)\right)^2 + i\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\Im(\alpha) - i\Im(\alpha)\Re(\alpha)\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{m\omega}{2\hbar}\left(x-\langle x\rangle_\alpha\right)^2 + \frac{i}{\hbar}\langle p\rangle_\alpha x - \frac{i}{2\hbar}\langle p\rangle_\alpha\langle x\rangle_\alpha\right). \qquad (23)
\end{aligned}$$

2 Time evolution

We can compute the time dependence of the coherent states by acting on $|\alpha\rangle$ by the time development operator:

$$U(t) = e^{-\frac{i}{\hbar}Ht}, \qquad (24)$$

which gives

$$|\alpha, t\rangle = U(t)|\alpha\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha\rangle, \qquad (25)$$

where $|\alpha, t\rangle$ is the time dependent coherent state. Expanding the coherent state in the energy eigenbasis, we get:

$$\begin{aligned}
|\alpha(t)\rangle &= U(t)|\alpha\rangle = e^{-\frac{i}{\hbar}Ht}|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-\frac{iHt}{\hbar}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(n+\frac{1}{2})}|n\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(n+\frac{1}{2})}\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{-i\frac{\omega t}{2}}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}a^\dagger\right)^n}{n!}|0\rangle \\
&= e^{-i\frac{\omega t}{2}}\left(e^{-\frac{1}{2}|\alpha|^2}e^{\alpha e^{-i\omega t}a^\dagger}|0\rangle\right) = e^{-i\frac{\omega t}{2}}|e^{-i\omega t}\alpha\rangle, \qquad (26)
\end{aligned}$$

where we used Eq. (17). We can drop the overall phase factor to write:

$$\alpha(t) = e^{-i\omega t}\alpha. \qquad (27)$$

In order to compute $\langle x\rangle_{\alpha(t)}$, let's first write $\alpha = |\alpha|e^{i\sigma}$, where σ is the initial phase. Then we have

$$\langle\hat{x}\rangle_{\alpha(t)} = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\Re(\alpha(t)) = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\sigma-\omega t). \qquad (28)$$

Similarly, the expected value of $\hat{p}$ reads:

$$\langle\hat{p}\rangle_{\alpha(t)} = \sqrt{m\hbar\omega}\sqrt{2}\Im(\alpha) = \sqrt{2m\hbar\omega}|\alpha|\sin(\sigma-\omega t). \qquad (29)$$

Equations (28) and (29) show that the expected values of x and p in the coherent state look just like the classical harmonic oscillator.

Finally, let's compute the explicit form of the wave function with its time dependence. Al we need to do is to transform $\alpha \to e^{-i\omega t}\alpha$ and $|\alpha\rangle \to e^{-i\frac{\omega t}{2}}|e^{-i\omega t}\alpha\rangle$ and put these back in Eq. (23)

$$\psi_\alpha(x,t) = \langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{m\omega}{2\hbar}\left(x-\langle x\rangle_{\alpha(t)}\right)^2 + \frac{i}{\hbar}\langle p\rangle_{\alpha(t)}x - i\frac{\omega t}{2} + i\frac{|\alpha|^2}{2}\sin\left(2\omega t - 2\sigma\right)\right). \quad (30)$$

[1] B. Bagchi, R. Ghosh, and A. Khare, "A pedestrian introduction to coherent and squeezed states," *International Journal of Modern Physics A*, vol. 35, no. 19, pp. 2030011–164, Jul. 2020, doi: 10.1142/S0217751X20300112. [Online]. Available: http://arxiv.org/abs/2004.08829

[2] I. Bialynicki-Birula, "Nonstandard Introduction to Squeezing of the Electromagnetic Field," *Acta Physica Polonica B*, vol. 29, no. 12, p. 3569, Dec. 1998 [Online]. Available: http://arxiv.org/abs/quant-ph/9809069