# Scattering fermions and scalars

**Abstract**

Some simple calculations on scalar-spinor scattering.

**Index Terms**

quantum,scattering,feynman diagrams, Yukawa

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## I. Lagrangian and Feynman Diagrams

We would like to compute the cross section of fermion-boson scattering process. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi + h\phi\bar{\psi}\psi - \frac{\lambda}{4!}\phi^4, \tag{1}$$

where $\phi$ represents the neutral scalar particle, and $\psi_\alpha$ is a four-component spinor field with $\alpha = 1, 2, 3, 4$. The scattering process we are after is given as

$$\phi(k_1)\psi(p_1) \longrightarrow \phi(k_2) + \psi(p_2). \tag{2}$$

The Fenynman diagrams contributing to the process are shown in Fig. 1.

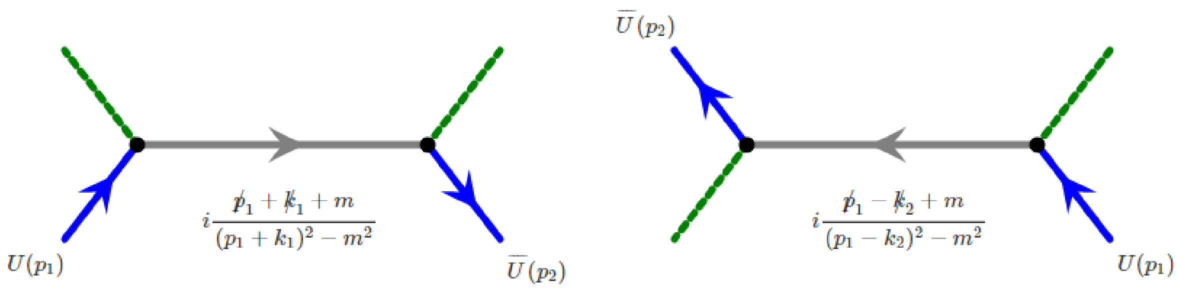


Figure 1: Two Feynman diagrams, with amplitudes $\mathcal{M}_A$ and $\mathcal{M}_B$, contributing to the scattering.

## II. Amplitudes

The amplitudes for the diagrams in Fig. 1 can be written as

$$\begin{aligned}\mathcal{M}_A &= -i\overline{U}(p_2)(-ih)\left[i\frac{\not p_1 + \not k_1 + M}{(p_1+k_1)^2 - M^2}\right](-ih)U(p_1)\\ \mathcal{M}_B &= -i\overline{U}(p_2)(-ih)\left[i\frac{\not p_1 - \not k_2 + M}{(p_1-k_2)^2 - M^2}\right](-ih)U(p_1).\end{aligned} \tag{3}$$

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The numerators can be simplified by using the equation of motion for the fermions, namely:

$$(\not{p}_1 - M)U(p_1) = 0. \tag{4}$$

Let's us compute the denominators :

$$\begin{aligned}(p_1+k_1)^2 - M^2 &= p_1^2 + k_1^2 + 2p_1\cdot k_1 - M^2 = M^2 + m^2 + 2p_1\cdot k_1 - M^2 \\ &= 2p_1\cdot k_1 + m^2 \\ (p_1-k_2)^2 - M^2 &= p_1^2 + k_2^2 - 2p_1\cdot k_2 - M^2 = M^2 + m^2 - 2p_1\cdot k_2 - M^2 \\ &= -2p_1\cdot k_2 + m^2. \end{aligned} \tag{5}$$

Inserting these into Eq. (3), we get

$$\begin{aligned}\mathcal{M}_A &= \frac{-h^2}{2p_1\cdot k_1 + m^2}\overline{U}(p_2)\left[2M + \not{k}_1\right]U(p_1) \\ \mathcal{M}_B &= \frac{h^2}{2p_1\cdot k_2 + m^2}\overline{U}(p_2)\left[2M - \not{k}_2\right]U(p_1). \end{aligned} \tag{6}$$

Let's also consider the process in the high energy limit, i.e., $E \gg M, m$, that is we will drop the mass terms. In this limit we can simplify the amplitudes:

$$\begin{aligned}\mathcal{M}_A &\simeq \frac{-h^2}{2p_1\cdot k_1}\overline{U}(p_2)\not{k}_1 U(p_1) \\ \mathcal{M}_B &\simeq \frac{-h^2}{2p_1\cdot k_2}\overline{U}(p_2)\not{k}_2 U(p_1). \end{aligned} \tag{7}$$

## III. Squaring the amplitudes

The total amplitude is given by

$$\mathcal{M} = \mathcal{M}_A + \mathcal{M}_B, \tag{8}$$

and we will need to compute its mode-square which will involve mode-squares of the individual amplitudes and the cross terms. We will also average over fermion polarization which will result in trace operations. There are few trace properties of $\gamma-$matrices we will make use of:

$$\mathrm{Tr}[I] = 4$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{9}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}] \tag{10}$$

$$\mathrm{Tr}[\gamma_1^\mu\gamma_2^\mu\cdots\gamma_{2n+1}^\mu] = 0, \tag{11}$$

The mode-square of the first amplitude becomes

$$\begin{aligned}\left|\overline{\mathcal{M}_A}\right|^2 &= \frac{h^4}{4(p_1\cdot k_1)^2}\frac{1}{2}\,\mathrm{Tr}\left[\not{p}_2\not{k}_1\not{p}_1\not{k}_1\right] \\ &= \frac{h^4}{2(p_1\cdot k_1)^2}p_1\cdot k_1\, p_2\cdot k_2 = h^4\frac{p_1\cdot k_2}{p_1\cdot k_1}. \end{aligned} \tag{12}$$

Similarly, the mode-square of the second amplitude reads

$$\begin{aligned}\left|\overline{\mathcal{M}_B}\right|^2 &= \frac{h^4}{4(p_1\cdot k_1)^2}\frac{1}{2}\,\mathrm{Tr}\left[\not{p}_2\not{k}_2\not{p}_1\not{k}_2\right] \\ &= \frac{h^4}{(p_1\cdot k_1)^2}p_2\cdot k_2\, p_1\cdot k_2 = h^4\frac{p_1\cdot k_1}{p_1\cdot k_2}, \end{aligned} \tag{13}$$

where we used conservation of $4-$momentum in the last step as follows:

$$\begin{aligned}p_1 + k_1 &= p_2 + k_2 \iff p_1 - k_2 = p_2 - k_1 \\ (p_1+k_1)^2 &= (p_2+k_2)^2 \implies p_1\cdot k_1 = p_2\cdot k_2, \end{aligned} \tag{14}$$

Finally one of the cross term can be calculated as

$$\begin{aligned}
\overline{\mathcal{M}_A^* \mathcal{M}_B} &= \frac{-h^4}{4 p_1 \cdot k_1 \, p_1 \cdot k_2} \frac{1}{2} \,\mathrm{Tr}\,[\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_2] \\
&= \frac{h^4}{2 p_1 \cdot k_1 \, p_1 \cdot k_2} \left[ p_2 \cdot k_1 \, p_1 \cdot k_2 + p_2 \cdot k_2 \, p_1 \cdot k_1 - p_2 \cdot p_1 \, k_1 \cdot k_2 \right] \\
&= \frac{h^4}{2} \left[ \frac{p_1 \cdot k_2}{p_1 \cdot k_1} + \frac{p_1 \cdot k_1}{p_1 \cdot k_2} - \frac{p_1 \cdot p_2 \, k_1 \cdot k_2}{p_1 \cdot k_1 \, p_1 \cdot k_2} \right]. \qquad (15)
\end{aligned}$$

## IV. Cross-section

Let's find out which term will have the dominant contribution to the cross-section. To this end, we can treat the problem in the center of mass frame and define:

$$\begin{aligned}
k_1 &= (\omega, 0, 0, w) \\
p_1 &= (E, 0, 0, -\omega) \\
k_2 &= (\omega, \omega \sin\theta, 0, \omega \cos\theta) \\
p_1 &= (E, 0, 0, -\omega). \qquad (16)
\end{aligned}$$

We can observe that the term $1/p_1 \cdot k_2$ will be $\sim 1/M^2$ at $\theta = \pm\pi$, and therefore will be the dominating term, since other terms will will behave as $1/E^2$. So the cross-section will be dominated by the following term

$$\frac{p_1 \cdot k_1}{p_1 \cdot k_2} = \frac{E + \omega}{E + \omega \cos\theta}. \qquad (17)$$

The differential cross-section becomes:

$$d\sigma = \frac{1}{2} \frac{1}{2} \frac{1}{2E} \frac{1}{2\omega} \frac{\omega}{8\pi} \frac{1}{E + \omega} 2h^4 \frac{E + \omega}{E + \omega \cos\theta} d\cos\theta, \qquad (18)$$

which is easily integrable to

$$\sigma = \frac{h^4}{16s} log \left( \frac{s}{M^2} \right), \qquad (19)$$

where $s \equiv (E + \omega)^2$.