Stirling approximation for factorial

Abstract

Stirling's approximation for the factorial function.

Index Terms

Thermodynamics, Math, factorial

Consider the following integral:

$$\int_0^\infty dx x^n e^{-x} = \left[(-1)^n \frac{d^n}{d\alpha^n} \int_0^\infty dx e^{-\alpha x}\right]_{\alpha=1} = \left[(-1)^n \frac{d^n}{d\alpha^n} \frac{1}{\alpha}\right]_{\alpha=1} = n!. \tag{1}$$

Taking this definition, we can do the following:

$$n! = \int_0^\infty dx x^n e^{-x} = \int_0^\infty dx e^{nln(x)-x}. \tag{2}$$

Let's take a close look at the function in the exponent:

$$u(x) \quad = \quad nln(x) - x, \tag{3}$$

as shown in Fig. 1. This function has its peak value at $x = n$. Note that this function appears in the exponent, under the integral. The dominant contribution to the integral will come from the domain around $x = n$. we can expand $u(x)$ around $x = n$:

$$\begin{aligned} u(x) &= nln(x) - x = nln(x - n + n) - x = nln(n[1 + \frac{x-n}{n}]) - x \\ &\simeq n\left(ln(n) + \frac{x-n}{n} - \frac{1}{2}\left[\frac{x-n}{n}\right]^2\right) - x = nln(n) - n - \frac{1}{2}\frac{(x-n)^2}{n} \equiv \tilde{u}(x). \end{aligned} \tag{4}$$

The original function and the approximated functions are plotted in Fig. 1.

n

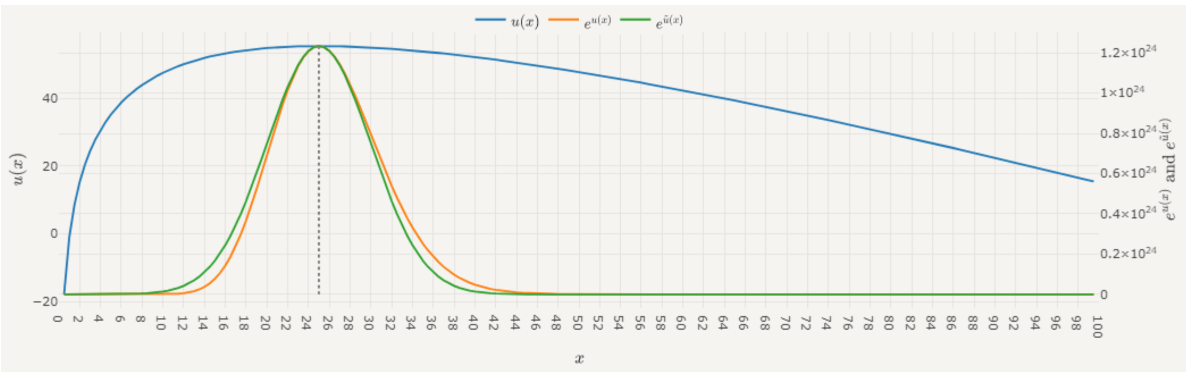

Figure 1: Interactive plot showing $u(x)$ (left), $e^{u(x)}$, and $e^{\tilde{u}(x)}$ (right).

From Fig. 1, we also notice that if we extended the x range to include negative values, the integral would not change much since $e^{\frac{1}{2}\frac{(x-n)^2}{n}}$ is rapidly decaying. Therefore we can change the lower limit of the integral from 0 to $-\infty$ to get:

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Find the interactive HTML-document here.

$$
\begin{aligned}
n! &= \int_0^\infty dx x^n e^{-x} = \int_0^\infty dx e^{u(x)} \simeq \int_0^\infty dx e^{\tilde{u}(x)} = n^n e^{-n} \int_0^\infty dx e^{-\frac{1}{2}\frac{(x-n)^2}{n}} \\
&\simeq n^n e^{-n} \int_{-\infty}^\infty dx e^{-\frac{1}{2}\frac{(x-n)^2}{n}} = n^n e^{-n}\sqrt{2\pi n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .
\end{aligned} \tag{5}
$$

Figure 2 shows the comparison of $n!$ with the Stirling's approximation given in Eq. (5).

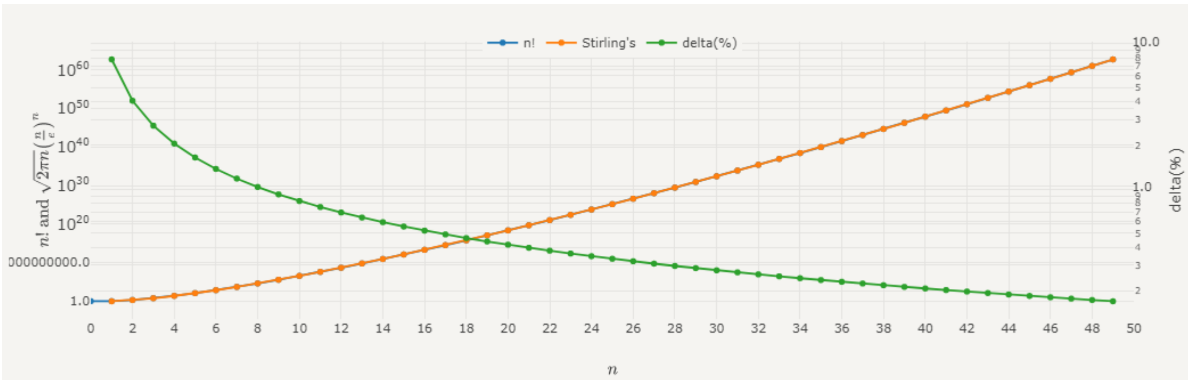

Figure 2: The plot of $n!$ and its Stirling's approximation (left). The relative error(right) gets smaller as n increases.