Musings on Weibull distribution

Abstract

This is a deep dive into Weibull distribution with an interactive visualization.

Index Terms

Reliability

I. Introduction

The Weibull distribution has two parameters: α parameter sets the time scale whereas the shape parameter β sets the shape of the distribution. The failure probability density, cumulative failure distributions and the hazard rate (failure rate) are defined as follows:

$$f_{\alpha,\beta}(t) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, \tag{1}$$

$$F_{\alpha,\beta}(t) = \int_0^t d\tau f_{\alpha,\beta}(\tau) = 1 - e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, \tag{2}$$

$$h_{\alpha,\beta}(t) = \frac{f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1}. \tag{3}$$

There is a misconception that Weibull distribution has a theoretical physical basis, which is clearly not true. In fact, Weibull was first to acknowledge that in the very original paper [1]:

> ... The objection has been stated that this distribution function has no theoretical basis. But in so far as the author understands, there are-with very few exceptions-the same objections against all other [functions], applied to real populations from natural or biological fields, at least in so far as the theoretical basis has anything to do with the population in question. Furthermore, it utterly hopeless to expect a theoretical basis for distribution functions of random variables...
>
> *Ernst Hjalmar Waloddi Weibull*

I think one of the root causes of this misconception is the way this topic is introduced. Typically, the functions are defined in the order as in Eqs. (1) -(3). $F_{\alpha,\beta}$, for example, is so neat and so simple, it almost looks like it comes off of a theoretical derivation. A more natural approach would have started from $h_{\alpha,\beta}$, i.e., the hazard rate. It is the failure rate normalized to the survivors. What can the simplest form of h be? Let's say it just a constant, and see where it gets us:

$$
\begin{aligned}
h_{\alpha,\beta}(t) \quad &\equiv \quad \lambda = \frac{f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} = \frac{\frac{d}{dt} F_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} = -\frac{d}{dt}\left[ln\left(1 - F_{\alpha,\beta}(t)\right)\right] \\
&\implies \quad F_{\alpha,\beta}(t) = 1 - e^{-\int_0^t d\tau \lambda} = 1 - e^{-\lambda t}.
\end{aligned}
\tag{4}
$$

There you have it: if you go with the simplest assumption on $h_{\alpha,\beta}$, you get the exponential distribution.

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Find the interactive HTML-document here.

Let's try a bit more generic function for h. Let's say it is a function of t, and try t^κ, where κ is a constant. It also needs a coefficient to fix the units, so let's try the following:

$$
\begin{aligned}
h_{\alpha,\beta}(t) \quad &\equiv \quad \zeta t^\kappa = -\frac{d}{dt}\left[ln\left(1 - F_{\alpha,\beta}(t)\right)\right] \equiv \lambda \\
&\implies \quad F_{\alpha,\beta}(t) = 1 - e^{-\zeta \int_0^t d\tau \tau^\kappa} = 1 - e^{-\frac{\zeta}{\kappa+1} t^{\kappa+1}}.
\end{aligned} \tag{5}
$$

Again, κ and ζ are some constants, however we do not know what they are. Just to simplify the expression we can redefine $\kappa + 1 \to \beta$, and and $\zeta \to \frac{\beta}{\alpha^\beta}$, which gives $F_{\alpha,\beta}(t) = 1 - e^{-\left(\frac{t}{\alpha}\right)^\beta}$.

The take away from this analysis is that, there is nothing mysterious about the Weibull function. It is basically the second simplest guess one can make.
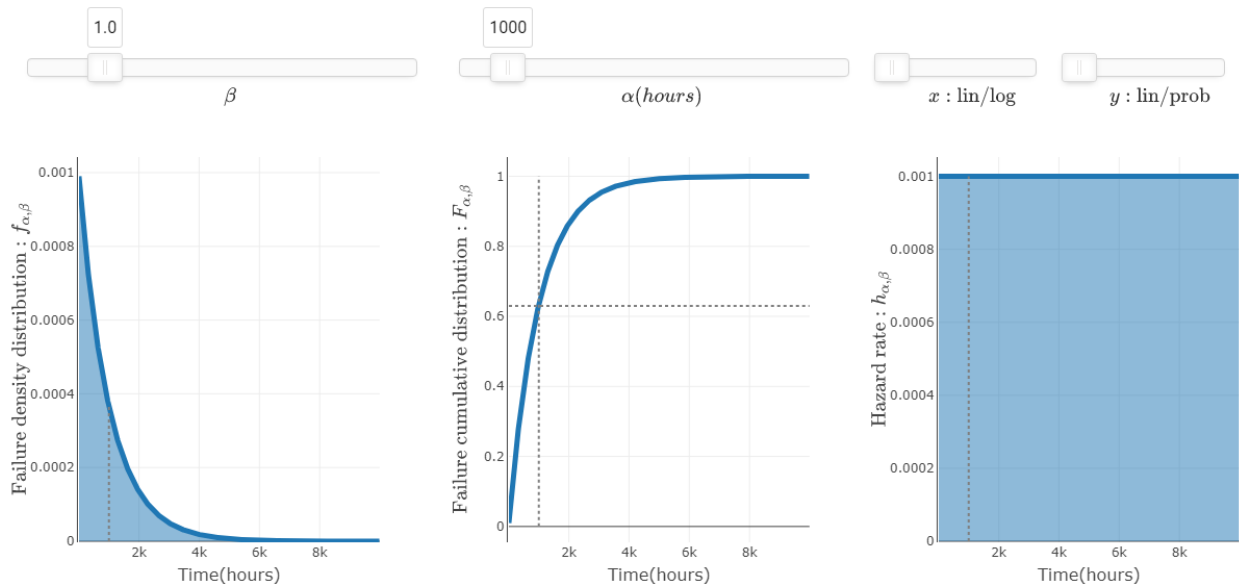
II. Visualizing failure distributions

Figure 1: A plot of Weibull distributions defined in Eqs.(1) -(3). This is a static copy, find the interactive HTML-document here.

[1] W. Weibull, "A statistical distribution function of wide applicability," *Journal of Applied Mechanics*, vol. 18, pp. 293–297, 1951 [Online]. Available: http://web.cecs.pdx.edu/cgshirl/Documents/Weibull-ASME-Paper-1951.pdf