# $L^2$ is a Hermitian operator

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A proof that square of the angular momentum vector is a Hermitian operator.

blog: https://tetraquark.vercel.app/posts/angular_momentum_squared/?src=pdf

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We need to show that

$$\int \psi_1^*(L^2\psi_2)d\Omega = \left[\int \psi_2^*(L^2\psi_1)d\Omega\right]^*, \tag{1}$$

where

$$L^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{2}$$

The second part of $L^2$ operator is easier to handle. The relevant part of the integral is the $\phi$ integral, which can be computed as follows

$$\begin{aligned}
\int_0^{2\pi} \psi_1^* \left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_2\right] d\phi &= \int_0^{2\pi} \frac{\partial}{\partial\phi}\left[\psi_1^*\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\psi_2\right] d\phi - \int_0^{2\pi}\left(\frac{\partial}{\partial\phi}\psi_1\right)^*\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\psi_2\right) d\phi \\
&= -\int_0^{2\pi} \frac{\partial}{\partial\phi}\left[\left(\frac{\partial}{\partial\phi}\psi_1\right)^* \frac{1}{\sin^2\theta}\psi_2\right] d\phi + \int_0^{2\pi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_1\right)^*\psi_2 d\phi \\
&= \left[\int_0^{2\pi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi_1\right)\psi_2^* d\phi\right]^*,
\end{aligned} \tag{3}$$

where we dropped the first terms in the first two lines as the they are the difference of the integrand at $\phi = 2\pi$ and $\phi = 0$, and that is zero as $\phi$ coordinate is $2\pi$ periodic.

The first part of $L^2$ operator seems to be harder because when we integrate by parts $\frac{\partial}{\partial\theta}$ will act on $\sin\theta$, which will complicate the problem. However, we can avoid it by a change of variable $u = \cos\theta$. The relevant part of the integral is the $dcos\theta$ integral, and with the above transformation it becomes,

$$
\begin{aligned}
\int_0^\pi \psi_1^* \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi_2\right) d\cos\theta &= \int_{-1}^1 \psi_1^* \frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_2\right) du \\
&= -\int_{-1}^1 \frac{\partial}{\partial u}\psi_1^* \left((1-u^2)\frac{\partial}{\partial u}\psi_2\right) du \\
&= -\int_{-1}^1 \left((1-u^2)\frac{\partial}{\partial u}\psi_1^*\right) \frac{\partial}{\partial u}\psi_2 du \\
&= \int_{-1}^1 \frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_1^*\right) \psi_2 du \\
&= \left[\int_{-1}^1 \psi_2^* \frac{\partial}{\partial u}\left((1-u^2)\frac{\partial}{\partial u}\psi_1\right) du\right]^* \\
&= \left[\int_0^\pi \psi_2^* \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\psi_1\right) d\cos\theta\right]^*, \qquad (4)
\end{aligned}
$$

where we dropped again some surface terms as $u^2 - 1 = 0$ at $u = \pm 1$. (If you prefer $\sin\theta d\theta$ integral instead of $dcos\theta$ integral, you will not need to change the variable.) This completes the proof that $L^2$ is Hermitian.

And that's it!