Biot-Savart Law

2025-01-03

This article presents a rigorous derivation of the Biot-Savart law from Maxwell's equations. Starting from the fundamental observation that magnetic fields are divergence-free, we develop the vector potential formulation and use Green's function methods to solve the resulting differential equation. The derivation demonstrates how the familiar Biot-Savart law emerges naturally from these first principles, providing both the general form for continuous current distributions and the specialized case for line currents. This treatment emphasizes the deep connection between the absence of magnetic monopoles and the mathematical structure of magnetic fields.

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Maxwell's Equations

From Maxwell's equations, we know that there are no magnetic monopoles. Therefore, the magnetic field is divergence free:

$$\nabla \cdot \mathbf{B} = 0. \tag{1}$$

This implies that the magnetic field can be written as the curl of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2}$$

We also know that the magnetic field is related to the current density by the following equation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{3}$$

Substituting Eq. 2 into Eq. 3 we get:

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}. \tag{4}$$

We can use the epsilon tensor to rewrite the left hand side of the equation:

$$(\nabla \times (\nabla \times \mathbf{A}))_i = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l A_m = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j\partial_l A_m = \partial_i \cancelto{0}{\nabla \cdot \mathbf{A}} - \nabla^2 \mathbf{A}_i. \quad (5)$$

Therefore, we have:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \quad (6)$$

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We can solve this equation by using the Green's function for the Laplacian. The Green's function for the Laplacian is given by:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \quad (7)$$

In order to convince ourselves that this is the correct Green's function, we can check that it satisfies the following equation:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \quad (8)$$

The defining property of the delta function is that it is zero everywhere except at the origin, where it is infinite. And also that the integral of the delta function over all space is one. Let's check $r \neq r'$ behaviour of the Green's function:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \nabla^2 \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi}\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\right) = \frac{1}{4\pi}\left(\frac{\nabla \cdot \mathbf{r}|\mathbf{r} - \mathbf{r}'|^3 - 3|\mathbf{r} - \mathbf{r}'|^3}{|\mathbf{r} - \mathbf{r}'|^6}\right) = 0 \quad (9)$$

Let's check the that it integrates to one:

$$\int d^3\mathbf{r} \nabla^2 G(\mathbf{r}, 0) = \int d^3\mathbf{r} \frac{1}{4\pi} \nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = \frac{1}{4\pi}\int d\mathbf{S} \cdot \mathbf{r}\frac{1}{r^3} = \frac{1}{4\pi}\int d\Omega = 1, \quad (10)$$

which confirms that the Green's function is correct. Inserting this into Eq. 6 we get:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad (11)$$

We will make use of the following vector identity:

$$\nabla \times \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \mathbf{J} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (12)$$

Once we have the vector potential, we can find the magnetic field by taking the curl of it:

$$\mathbf{B}(\mathbf{r}) \quad = \quad \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{(\mathbf{r} - \mathbf{r}')^3}, \tag{13}$$

which is the Biot-Savart law.

When the current is confined to a compact line, as in Figure 1, the volume integral can be replaced by a line integral:

$$\mathbf{B}(\mathbf{r}) \quad = \quad \frac{\mu_0 I}{4\pi} \int \frac{d \quad \times (\mathbf{r} - \mathbf{r}')}{(\mathbf{r} - \mathbf{r}')^3}. \tag{14}$$

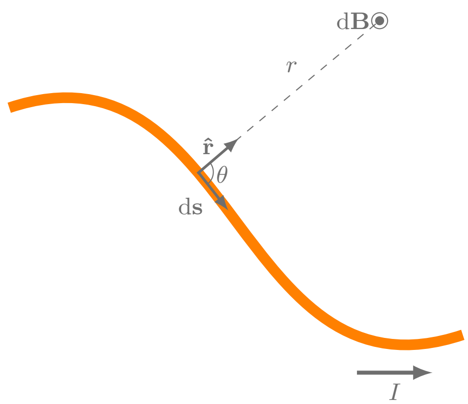

Figure 1: A segment of a wire carrying current I.