The catenary curve with a sliding end

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We dive into calculus of variations to calculate the shape of a rope fixed in one end and free to slide on the other. Starting with the classic catenary problem, we derive the Euler-Lagrange equation and show how the familiar hyperbolic cosine solution emerges from energy minimization principles. The real challenge comes when one end is free to slide along a vertical post, introducing movable boundary conditions.

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Introduction

You must have seen ropes and chains hanging from two posts. Ever wondered what kind of shape they take? A first guess would be a parabola or a higher order polynomial, but that would be wrong. It turns out to be a cosh scaled properly to pass through the end points and have the correct length. The rope sags into a shape that minimizes its total potential energy. **Calculus of variations** [1] is the study of such problems, and it forms the mathematical foundations of classical and modern physics.

This would have been one of many posts you would find elsewhere if the title did not include "with a sliding end". What we want to describe is a case where one end of the rope is anchored, and the other end is free to slide down the post. You can think of it being tied to a ring that can freely slide up and down. Such problems are known as variational problems with movable boundaries, and they are much richer than plain-old-fixed-end problems. I will set the stage by starting with the fixed ends case, and later move to the sliding-end problem. Just for entertainment, I also built a simple jig to physically verify that this is not just mathematical wizardry.

Functionals

A functional can be considered as an operation that takes in a function and returns a number. The most familiar functional is integration with fixed limits. It takes in $f(x)$ and spits out $\int_a^b f(x)dx$, which is just a number. Integration happens naturally in physics. For example, consider a rope hanging from two anchor points as illustrated in Figure 1

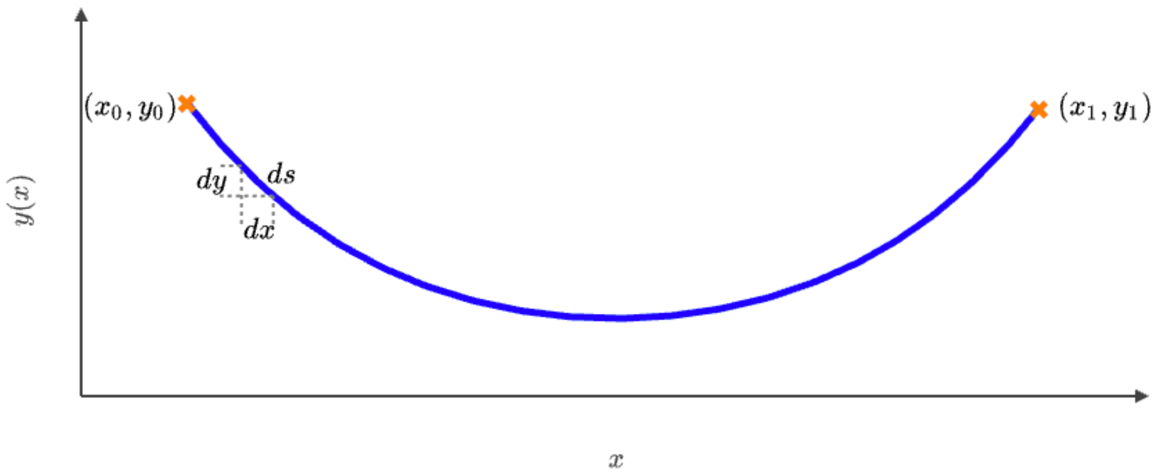

Figure 1: The orange curve $y(x)$, which is unknown at the moment, gives the minimum value for the functional $\mathscr{S}$. The green curve represents the new curve with random deformations around $y(x)$. The variation $\eta(x)$ must vanish at the end points since the values of q are fixed at these points.

The potential energy of the differential length ds is $dg\,y(x)ds$, where d is the mass density of the rope, and g is the gravitational acceleration. We can compute the potential energy by integrating over the length to get v=d g $\propto \int_a^b y(x)ds$, where $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}dx$. This is also how $y' \equiv \frac{\partial y}{dx}$ naturally shows up is such problems.

Variational calculus

In a generic case we will have the functional v of this form:

$$v = \int_{x_0}^{x_1} \mathscr{L}(x, y, y')dx, \tag{1}$$

where $\mathscr{L}$ is the function of interest [more on that later]. Let's assume that we have a function $y(x)$ that gives the minimum value for v. If we fiddle y around the optimal function by a small amount $\alpha\eta(x)$, i.e., $y(x) \to y(x) + \alpha\eta(x)$, where $\eta(x)$ is an arbitrary function and α is a small number, then the change in v should be 0. This is analogous to requiring that the derivative should vanish at a local extremum of the function, that is: $\frac{df(x)}{dx}|_{x=x^*} = 0$. Rigorously speaking, we can define the following functional

$$v(\alpha) = \int_{x_0(\alpha)}^{x_1(\alpha)} \mathscr{L}(x, y + \alpha\eta, y' + \alpha\eta')dx, \tag{2}$$

and require that

$$\left.\frac{dv(\alpha)}{d\alpha}\right|_{\alpha=0} = 0. \tag{3}$$

How we will proceed will depend on the conditions we impose that the end points x_0 and x_1.

Both ends fixed

Consider a problem where the end points are specified. This implies that we are not free to wiggle y at the end points x_0 and x_1, i.e.,

$$\eta(x_0) = \eta(x_1) = 0. \tag{4}$$

The variation is illustrated in Figure 2.

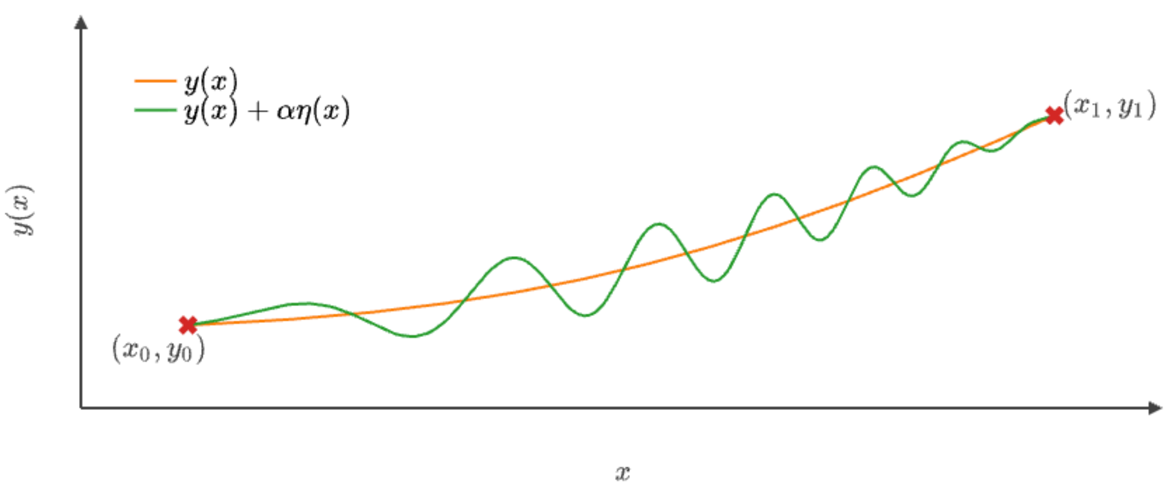

Figure 2: The orange curve $y(x)$, which is unknown at the moment, gives the minimum value for the functional $\mathscr{S}$. The green curve represents the new curve with random deformations around $y(x)$. The variation $\eta(x)$ must vanish at the end points since the values of q are fixed at these points.

Keeping the boundary conditions in Eq. 4 in mind, let us calculate Eq. 3:

$$
\begin{aligned}
\left.\frac{dv(\alpha)}{d\alpha}\right|_{\alpha=0} &= \int_{x_0}^{x_1} \left.\frac{d}{d\alpha}\mathscr{L}(x, y+\alpha\eta, y'+\alpha\eta'(x))\right|_{\alpha=0} dx \\
&= \int_{x_0}^{x_1} \left[\frac{\partial}{\partial y}\mathscr{L}(x,y,y')\eta + \frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\frac{d\eta}{dx}\right] dx \\
&= \int_{x_0}^{x_1} \left[\frac{\partial}{\partial y}\mathscr{L}(x,y,y')\eta + \frac{d}{dx}\left(\frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\eta\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\right)\eta\right] dx \\
&= \int_{x_0}^{x_1} \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y'}\right)\right]\eta dx + \left.\frac{\partial \mathscr{L}}{\partial y'}\eta\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \left[\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y'}\right)\right]\eta dx, \quad (5)
\end{aligned}
$$

where the boundary terms become 0 due to the constraints in Eq. 4. Since η is an arbitrary function, in order to set this equation to 0, we require the following:

$$
\frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathscr{L}}{\partial y'}\right) = 0. \tag{6}
$$

Equation 6 is known as the Euler-Lagrange equation and the function $\mathscr{L}$ is called the Lagrangian. In the case of the hanging rope, we have

$$
\mathscr{L} = dgy\sqrt{1+y'^2}. \tag{7}
$$

We can plug this into Eq. 6 and solve the resulting differential equation for $y(x)$. However, it is easy to see that it will be a second order differential equation. It won't be too hard to solve, but we can do better than that. The crucial observation is that $\mathscr{L}$ has no explicit x dependence, i.e., $\frac{\partial \mathscr{L}}{\partial x} = 0$. This means the total derivative of $\mathscr{L}$ can be written as:

$$
\frac{d\mathscr{L}}{dx} = \frac{\partial \mathscr{L}}{\partial x} + \frac{\partial \mathscr{L}}{\partial y}y' + \frac{\partial \mathscr{L}}{\partial y'}y'' = \cancel{\frac{\partial \mathscr{L}}{\partial x}} + \frac{\partial \mathscr{L}}{\partial y}y' + \frac{\partial \mathscr{L}}{\partial y'}y'' = \frac{\partial \mathscr{L}}{\partial y}y' + \frac{\partial \mathscr{L}}{\partial y'}y''. \tag{8}
$$

We can create these terms out of the Eq. 6 if we multiply it with y':

$$
\frac{d\mathscr{L}}{\partial y}y' - \frac{d}{dx}\left(\frac{d\mathscr{L}}{\partial y'}\right)y' \quad = \quad \frac{d\mathscr{L}}{\partial y}y' - \frac{d}{dx}\left(\frac{d\mathscr{L}}{\partial y'}y'\right) + \frac{d\mathscr{L}}{\partial y''}y'' = \frac{d}{dx}\left(\mathscr{L} - \frac{d\mathscr{L}}{\partial y'}y'\right) = 0, \tag{9}
$$

which means that

$$
\mathscr{L} - \frac{d\mathscr{L}}{\partial y'}y' = C. \tag{10}
$$

This reduces the order of the differential equation from two to one! Inserting the expression for $\mathscr{L}$ into Eq. 7 we get:

$$
y\sqrt{1+y'^2} - \frac{yy'2}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}} = C. \tag{11}
$$

It is best to solve the equation in a parametric form by defining $y' = \sinh t$

$$y = C\sqrt{1+y'^2} = C\sqrt{1+\sinh^2 t} = C\cosh t. \tag{12}$$

We can extract $x(t)$ as follows:

$$\frac{dy}{dt} = C\sinh t = \frac{dy}{dx}\frac{dx}{dt} = \sinh t\frac{dx}{dt} \to \frac{dx}{dt} = C, \tag{13}$$

which gives $x = Ct + D$. We can eliminate t in favor of x: $t = \frac{x-D}{C}$ and put it back it $y(t)$ to get

$$y(x) = C\cosh\left(\frac{x-D}{C}\right). \tag{14}$$

C and D are the integration constants and they can be fixed by requiring that $y(x_0) = y_0$ and $y(x_0) = y_1$, i.e., the anchor points are fixed.

If you think about this practically, you will notice a problem. You have a rope and you can decide on the anchor points as you wish. That completely fixes all the constants. How about the length of the rope, though? A longer rope will definitely have a different shape than that of a shorter one. There should have been another parameter in our solution so that it can be adjusted to give the correct length. That is why we have to introduce a *Lagrange multiplier* to address variation problems with constraints. In this case the constraint is that the solution should give the correct length: $\int_{x_0}^{x_0} ds = L$, L being the length of the rope. In order to enforce this requirement we revise $\mathscr{L}$ to $\mathscr{L} - \lambda\left(\int_{x_0}^{x_0} ds - L\right)$ where λ is the Lagrange parameter. The new Lagrangian can be written as[1]

$$\mathscr{L} = dg(y-\lambda)\sqrt{1+y'^2}, \tag{15}$$

Note that we don't have to solve the differential equation all over again since the new term just shifts y. Therefore, the final solution is simply a shifted version of previous one:

$$y(x) = \lambda + C\cosh\left(\frac{x-D}{C}\right). \tag{16}$$

This makes more sense now: we have a solution with 3 free parameters and we have 3 conditions [2 end points and the length]. Imposing the conditions we will get a unique solution. Let's do that by first calculating the length:

$$\begin{aligned} L &= \int_{x_0}^{x_1} ds = \int_{x_0}^{x_1} \sqrt{1+y'^2}dx = \int_{x_0}^{x_1} \sqrt{1+\sinh^2\left(\frac{x-D}{C}\right)}dx = \int_{x_0}^{x_1} \cosh\left(\frac{x-D}{C}\right)dx \\ &= C\left[\sinh\left(\frac{x_1-D}{C}\right) - \sinh\left(\frac{x_0-D}{C}\right)\right]. \end{aligned} \tag{17}$$

[1]We absorb the prefactor gd by redefining $\frac{\lambda}{gd}$ as λ.

Along with the end point requirements, we have the following conditions:

$$\begin{aligned} y(x_0) = \lambda + C\cosh\left(\frac{x_0 - D}{C}\right) &\equiv y_0 \\ y(x_1) = \lambda + C\cosh\left(\frac{x_1 - D}{C}\right) &\equiv y_1 \\ C\left[\sinh\left(\frac{x_1 - D}{C}\right) - \sinh\left(\frac{x_0 - D}{C}\right)\right] &= L. \end{aligned} \tag{18}$$

In principle, these equations can be solved numerically, but we can simplify them a bit. Taking the difference of two lines gives:

$$\frac{y_1 - y_0}{C} = \cosh\left(\frac{x_1 - D}{C}\right) - \cosh\left(\frac{x_0 - D}{C}\right). \tag{19}$$

Take its square and subtract the square of the third line:

$$\begin{aligned} \frac{(y_1 - y_0)^2}{C^2} - \frac{L^2}{C^2} &= \cosh^2\left(\frac{x_1 - D}{C}\right) - 2\cosh\left(\frac{x_1 - D}{C}\right)\cosh\left(\frac{x_0 - D}{C}\right) + \cosh^2\left(\frac{x_1 - D}{C}\right) \\ &\quad - \sinh^2\left(\frac{x_0 - D}{C}\right) + 2\sinh\left(\frac{x_0 - D}{C}\right)\sinh\left(\frac{x_0 - D}{C}\right) - \sinh^2\left(\frac{x_0 - D}{C}\right) \\ &= 2\left[1 - \cosh\left(\frac{x_0 - x_1}{C}\right)\right], \end{aligned} \tag{20}$$

which still needs to be solved numerically. What we accomplished by going through the algebra was that we reduced the problem from three equation with three unknowns to one equation with one unknown, C. Once we solve for C, we can insert it back into Eq. 19 to get D, and finally we can compute λ.

One end sliding

In many physical problems the end points might be movable. Consider a case where one end is fixed and the other one is not, as in Figure 3

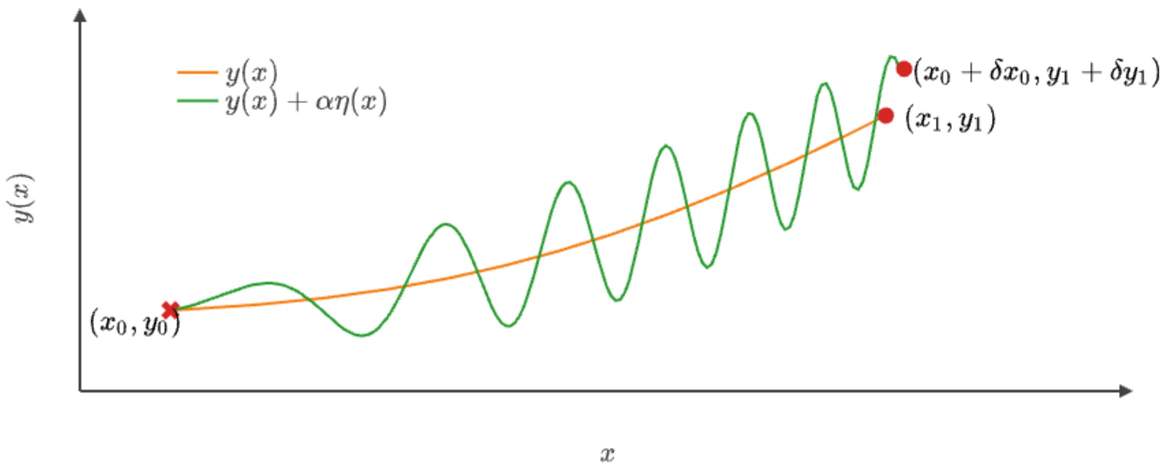

Figure 3: The orange curve $y(x)$, which is unknown at the moment, gives the minimum value for the functional $\mathscr{S}$. The green curve represents the new curve with random deformations around $y(x)$. The variation $\eta(x)$ must vanish at the end points since the values of q are fixed at these points.

This will require an important revision in the derivation of the Euler-Lagrange equation: we will not be able to drop the boundary terms. In order to improve the notation let us define the wiggle function $\alpha\eta(x)$ as δy. Now, we not only perturb y as $y + \delta y$ but also the boundary x_1 as $x_1 + \delta x_1$. With the perturbed paths and the perturbed boundary, the change in the functional can be written as

$$\begin{aligned}\delta v &= \int_{x_0}^{x_1+\delta x_1} \mathscr{L}(x, y+\delta y, y'+\delta y')dx - \int_{x_0}^{x_1} \mathscr{L}(x,y,y')dx \\ &= \int_{x_1}^{x_1+\delta x_1} \mathscr{L}(x, y+\delta y, y'+\delta y')dx + \int_{x_0}^{x_1} \mathscr{L}(x, y+\delta y, y'+\delta y')dx \\ &\quad - \int_{x_0}^{x_1} \mathscr{L}(x,y,y')dx, \end{aligned} \tag{21}$$

where we split the first integral into two pieces. Note that the range of the first integral is infinitesimally small, therefore we can simply take the value of the integrand and multiply if by the width, which is δx_1. The rest of the calculation is almost identical to what we did earlier with one difference: we dropped both of the boundary terms earlier and we can do that no more! We have to keep the upper one in this case since it is not necessarily zero. Then the variation becomes:

$$\begin{aligned}
\delta v &= \mathscr{L}\Big|_{x_1}\delta x_1 + \int_{x_0}^{x_1}\left[\frac{\partial}{\partial y}\mathscr{L}(x,y,y')\delta y + \frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\delta y'\right]dx \\
&= \mathscr{L}\Big|_{x_1}\delta x_1 \\
&\quad + \int_{x_0}^{x_1}\left[\frac{\partial}{\partial y}\mathscr{L}(x,y,y')\delta y + \frac{d}{dx}\left(\frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\delta y\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\mathscr{L}(x,y,y')\right)\delta y\right]dx \\
&= \mathscr{L}\Big|_{x_1}\delta x_1 + \frac{\partial\mathscr{L}}{\partial y'}\delta y\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}\left[\frac{\partial\mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\mathscr{L}}{\partial y'}\right)\right]\delta y dx \\
&= \mathscr{L}\Big|_{x_1}\delta x_1 + \frac{\partial\mathscr{L}}{\partial y'}\delta y\Big|_{x_1} + \int_{x_0}^{x_1}\left[\frac{\partial\mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\mathscr{L}}{\partial y'}\right)\right]\delta y dx. \qquad (22)
\end{aligned}$$

Since δy is an arbitrary function, we still require the following:

$$\frac{\partial\mathscr{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\mathscr{L}}{\partial y'}\right) = 0. \qquad (23)$$

In addition to that, we need:

$$\mathscr{L}\Big|_{x_1}\delta x_1 + \left[\frac{\partial\mathscr{L}}{\partial y'}\delta y\right]_{x_1} = 0. \qquad (24)$$

We should clearly state what $\delta y\Big|_{x_1}$ means: it is the vertical displacement at $x = x_1$, as illustrated in Figure 4.

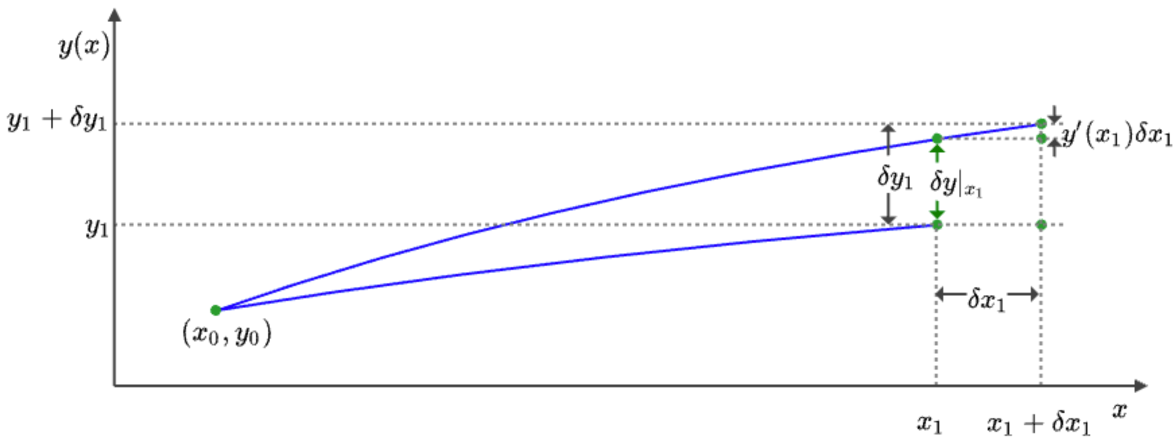

Figure 4: The orange curve $y(x)$, which is unknown at the moment, gives the minimum value for the functional $\mathscr{S}$. The green curve represents the new curve with random deformations around $y(x)$. The variation $\eta(x)$ must vanish at the end points since the values of q are fixed at these points.

As seen from the geometry in Figure 4, we can write the following equation

$$\delta y\Big|_{x_1} = \delta y_1 - y'(x_1)\delta x_1. \tag{25}$$

Putting this back in Eq. 24 we get

$$\left[\mathscr{L} - y'\frac{\partial \mathscr{L}}{\partial y'}\right]_{x_1} \delta x_1 + \frac{\partial \mathscr{L}}{\partial y'}\Big|_{x_1} \delta y_1 = 0. \tag{26}$$

If δx_1 and δy_1 are independent, we need to set the two terms in Eq. 26 to 0 individually. However, in most physical problems, the end point is constrained to move on a curve $y_1 = \varphi(x_1)$. In such cases we will have $\frac{\delta y_1}{\delta x_1} = \varphi'(x_1)$, and Eq. 26 simplifies to

$$\left[\mathscr{L} + (\varphi' - y')\frac{\partial \mathscr{L}}{\partial y'}\right]_{x_1} \delta x_1 = 0, \tag{27}$$

which is known as the **transversality condition.** For the specific case of $\mathscr{L}$ in Eq. 15 we get

$$\left[\frac{(y-\lambda)(1+\varphi' y')}{\sqrt{1+y'^2}}\right]_{x_1} = 0. \tag{28}$$

Assuming $y - \lambda \neq 0$ at the boundary, the only way to satisfy this equation would require $\varphi' y'|_{x_1} = -1$, that is y should be orthogonal to the curve φ. This is really neat. It all boils down this: the curve will still be a catenary, but when it hits the boundary, it should be perpendicular to it. For example, if you have a vertical post, the chain will be parallel to the ground at the post!

Neural Networks

Why on earth will you want to solve a differential equation with neural networks(NNs)? The first answer is, because why not if you can and you like applying NNs on everything?

Figure 5: The official slogan of the hot sauce manufacturer, Frank's Red Hot. Neural Networks are very powerfull and find applications in a wide range of problems.

A more reasonable answer would be that sometimes you have the differential equation that encodes the physics of the problem, and empirical data you collected. You want to blend these in to get the best solution. This is referred to as physics informed neural networks(PINN) [2], [3].

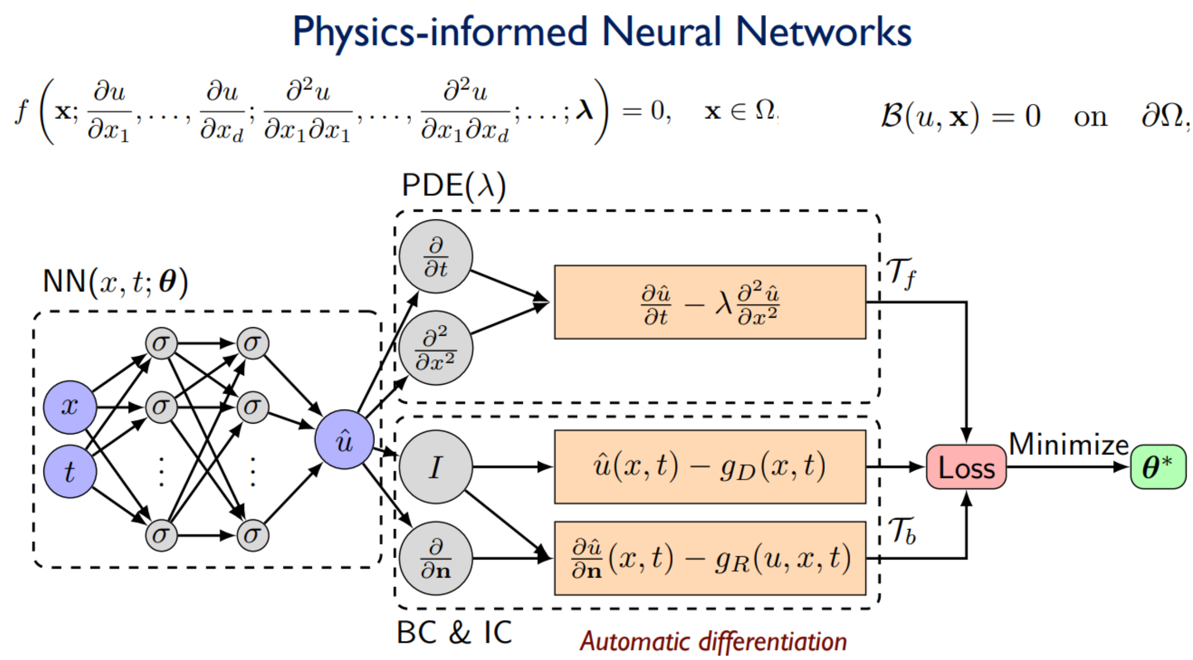

Figure 6: Illustration of a PINN. The loss function includes the deviation from the differential equation, the boundary conditions, and possibly empirical data. Credit Paris Perdikaris.

In this approach, the deviation from the boundary and the initial conditions are integrated into the loss function. During the training process, the network optimizes the parameters so that the approximate solution satisfies the differential equation and the boundary conditions, and -if available- the data, with the least amount of error. The method ensures that the final solution is in compliance with the differential equation, which stems from the underlying physical theory, hence the name physics informed neural network.

To be absolutely clear, for this particular problem, there is no reason to solve this problem with NN other than that it is fun. Let's do that and solve Eq. 10.

See the online version for the code.

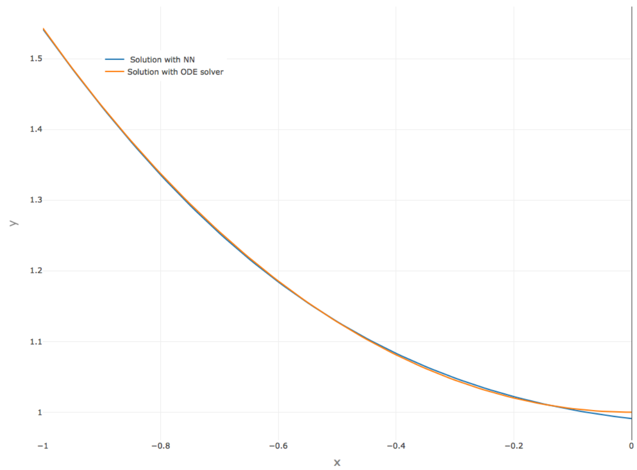

Figure 7: The catenary curve obtained with neural networks and with the regular ordinary differential equation (ODE) solver.

Figure 7 shows that it is indeed possible to solve a differential equation with neural networks.

Experimental tests

Does this work in real life? Can we confirm that ropes and chains indeed take the catenary shape? Let's take a look at the image taken at the beautiful city of Estes Park, CO, and see what it tells us.

Image processing with Python

This is not that much of a processing: we just want to load the image an overlay a cosh curve to see if it fits the rope shape.

See the online version for the code.

Figure 8: Drawing a
cosh curve onto the hanging rope in the original image results in a very good fit.

A test set up

How about the sliding end case? We probably won't find this out in the wild, so I have to build a test rig. I have a pile of metal shafts that I have been pulling off from dead printers. They are very polished and have low friction. I also found a plastic cylinder that fits perfectly on the shaft, and it slides easily. The curve indeed hits the shaft at 90°, and I find it very cool!

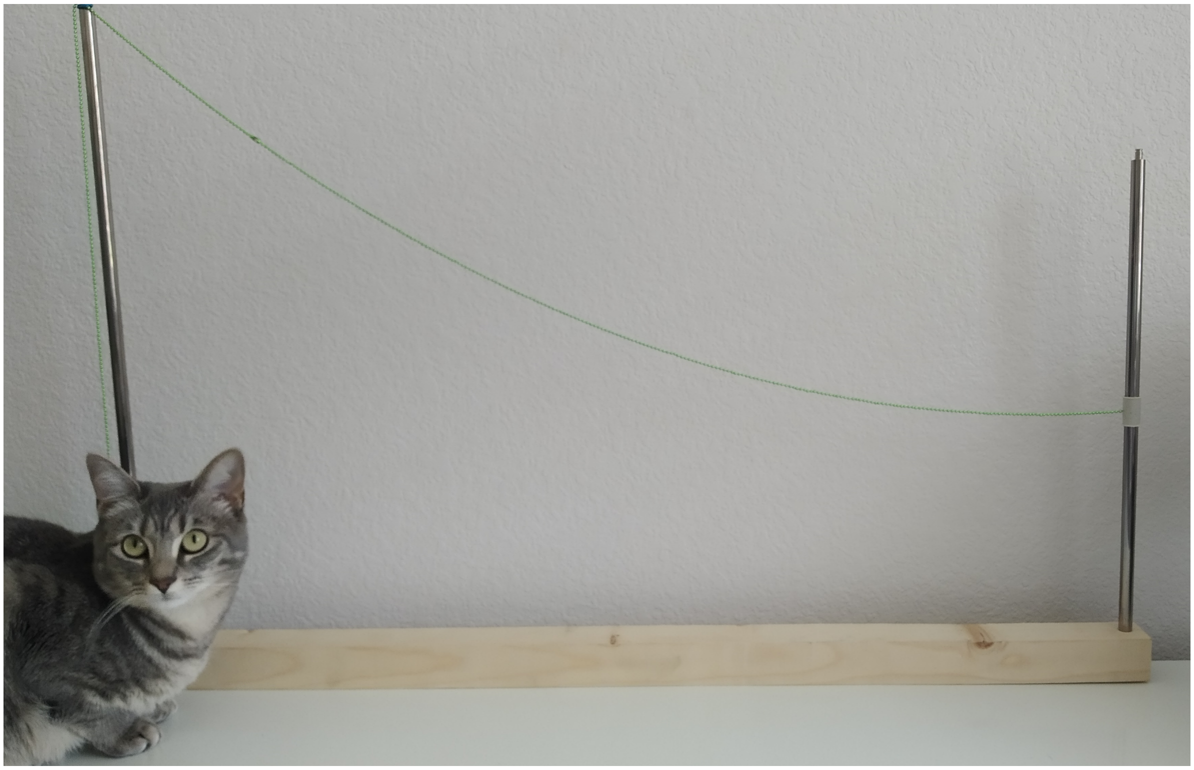

Figure 9: If one of the ends can freely slide, it will settle down to the point for which the chain leaves from the shaft at an angle of 90°. The curious cat is for scale.

Build your own catenary

Find the interactive catenary curve simulation with sliding mode here.

[1] L. D. Elsgolc, *Calculus of variations*. Dover Publications, 2007.

[2] M. Raissi, P. Perdikaris, and G. E. Karniadakis, "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations," *Journal of Computational Physics*, vol. 378, pp. 686–707, 2019.

[3] Maziar. Raissi, "Physics informed neural networks," *GitHub repository*. https://github.com/maziarraissi/PINNs; GitHub, 2020.