Integral of the month: $\int \frac{x^\alpha dx}{x^2 - 2\beta x + 1}$

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This article presents a fun exercise in complex analysis by evaluating the integral $\int_0^\infty \frac{x^\alpha dx}{x^2-2\beta x+1}$ using residue calculus. We explore the keyhole contour method to handle the branch cut created by the non-integer power x^α, and derive closed-form solutions for various parameter ranges. The approach demonstrates the elegance of complex integration techniques in solving seemingly difficult real integrals, making it an excellent pedagogical example for students learning residue theory and contour integration.

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The domain of convergence

We want to compute the integral $I = \int_0^\infty dx \frac{x^\alpha}{x^2-2\beta x+1}$ for a range of real-valued parameters α and β. Since the denominator is quadratic, we need to have $\alpha < 1$ so that the integral converges. Additionally, if α is an integer, the integral can be evaluated by partial fractions. Therefore, we will assume that α is not an integer. Furthermore, in order for the integral to converge, we also require $-1 < \alpha$. The other thing we have to check is the poles of the denominator. We first upgrade the real-valued parameter x to a complex number z, and define $f(z)$:

$$f(z) \equiv \frac{z^\alpha}{z^2 - 2\beta z + 1} = \frac{z^\alpha}{(z-z_1)(z-z_2)} \tag{1}$$

where $z_{1,2} = \beta \pm \sqrt{\beta^2 - 1}$ as shown in **?@fig-roots**.

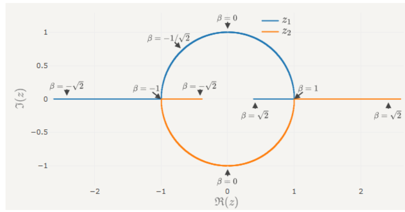

Figure 1: figcapPDF

If the roots fall on the positive real x-axis, the integral will diverge. From the plot we observe that if $\beta < 1$, the roots will not be on the positive x-axis. Therefore, the integral will be well defined for $\beta < 1$ and $-1 < \alpha < 1$.

The key-hole contour

Due to the x^α term with non-integer α, the integral has a branch cut. We can take the positive x-axis as the cut as shown in Figure 2.

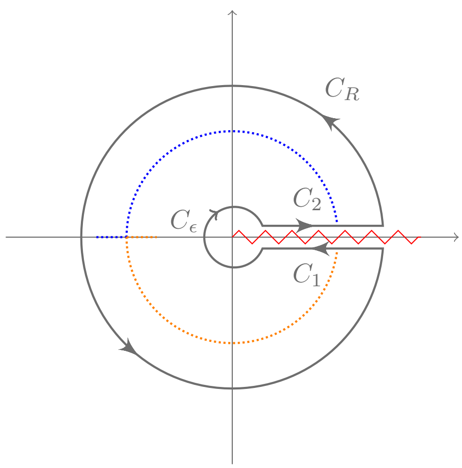

Figure 2: Key-hole contour to evaluate the integral. The dashed lines show the possible positions of the two poles.

Using the residue theorem, we can write:

$$\begin{aligned}\oint f(z)dz &= 2\pi i \left(\text{Res}(f, z_1) + \text{Res}(f, z_2)\right) = 2\pi i \left(\frac{z_1^\alpha}{z_1 - z_2} + \frac{z_2^\alpha}{z_2 - z_1}\right) \\ &= \frac{\pi i}{\sqrt{\beta^2 - 1}} \left[\left(\beta + \sqrt{\beta^2 - 1}\right)^\alpha - \left(\beta - \sqrt{\beta^2 - 1}\right)^\alpha\right] \qquad (2)\end{aligned}$$

On the left-hand side, the integrals over the circles C_R and C_ϵ vanish. We just need to figure out what happens on $C_{1,2}$. The integral on C_2 is the original integral we are looking to solve. The one on C_1 is

$$\int_{C_1} dz f(z) = \int_{C_1} dx \frac{x^\alpha e^{i2\pi\alpha}}{x^2 - 2\beta x + 1} = -\int_\epsilon^\infty \frac{x^\alpha e^{i2\pi\alpha}}{x^2 - 2\beta x + 1} = -e^{i2\pi\alpha} I \tag{3}$$

Therefore, the final result is

$$I = \frac{\pi i}{\sqrt{\beta^2 - 1}(1 - e^{i2\pi\alpha})} \left[\left(\beta + \sqrt{\beta^2 - 1}\right)^\alpha - \left(\beta - \sqrt{\beta^2 - 1}\right)^\alpha \right] \tag{4}$$

Various interesting cases

Let us look at a few specific cases.

$\beta = 0$ case

The roots are $z_{1,2} = \pm i$. The corresponding integral becomes:

$$\begin{aligned} I &= \int_0^\infty dx \frac{x^\alpha}{x^2 + 1} = \frac{\pi i}{i(1 - e^{i2\pi\alpha})} \left[i^\alpha - (-i)^\alpha \right] = \frac{\pi}{1 - e^{2\pi i\alpha}} \left[e^{i\pi\alpha/2} - e^{3\pi i\alpha/2} \right] \\ &= \frac{\pi}{e^{-i\pi\alpha} - e^{i\pi\alpha}} \left[e^{-i\pi\alpha/2} - e^{i\pi\alpha/2} \right] = \frac{\pi \sin(\pi\alpha/2)}{\sin(\pi\alpha)} = \frac{\pi}{2\cos(\pi\alpha/2)} \end{aligned} \tag{5}$$

$\beta = -1/\sqrt{2}$ case

The roots are $\{z_1, z_2\} = \{e^{3\pi i/4}, e^{5\pi i/4}\}$, and $\sqrt{\beta^2 - 1} = 1/\sqrt{2}$ The corresponding integral reads:

$$I = \int_0^\infty dx \frac{x^\alpha}{x^2 + \sqrt{2}x + 1} = \frac{i\pi\sqrt{2}}{(1 - e^{2\pi i\alpha})} \left[e^{3\pi i\alpha/4} - e^{5\pi i\alpha/4} \right] = \frac{\sqrt{2}\pi \sin(\pi\alpha/4)}{\sin(\pi\alpha)} \tag{6}$$

$\beta = -1$ case

This is a tricky case since the roots merge. We can either fall back onto the computation of residues with higher order poles, or we can simply approach this limit carefully by setting $\beta = -1 + \epsilon$ to get $z_{1,2} = -1 \pm \delta$ where $\delta \equiv \sqrt{2\epsilon}$ is a small positive number. Equivalently, $\{z_1, z_2\} = \{-e^{-i\delta}, -e^{i\delta}\}$ and $\sqrt{\beta^2 - 1} = \delta$. Then the integral becomes:

$$I = \int_0^\infty dx \frac{x^\alpha}{x^2 + 2x + 1} = \frac{\pi i}{\delta(1 - e^{2\pi i\alpha})} e^{i\pi\alpha} \left[e^{-i\delta\alpha} - e^{i\delta\alpha}\right] = \frac{\pi\alpha}{\sin(\pi\alpha)} \tag{7}$$

Putting it all together

Note that the complete answer is already given in Eq.4. One could simply plug in numbers and get the answer. However, it requires surgical precision to compute the function due to the branch cut: if one is not careful enough, they will cross the cut, and the result will be messed up due to the multi-valued nature of the functions. So let's dive into the expression in Eq.4 and simplify it very carefully.

$0 \leq \beta < 1$ case

In this range of β we will have $z_1 = \beta + i\sqrt{1-\beta^2} \equiv e^{i\theta}$ where $\theta = \arctan\left[\frac{\sqrt{1-\beta^2}}{\beta}\right]$, and $z_2 = \beta - i\sqrt{1-\beta^2} \equiv e^{2\pi i - i\theta}$. z_1 is in the first quadrant and z_2 is in the fourth. Note that we defined the angle of z_2 so that we don't cross the branch cut. We can write I as

$$\begin{aligned} I &= \frac{\pi i}{i\sqrt{1-\beta^2}(1-e^{i2\pi\alpha})} \left[e^{i\theta\alpha} - e^{2\pi\alpha i - i\theta\alpha}\right] = \frac{\pi}{\sqrt{1-\beta^2}} \frac{\sin\left[\alpha(\pi - \theta)\right]}{\sin(\pi\alpha)} \\ &= \frac{\pi}{\sqrt{1-\beta^2}} \frac{\sin\left\{\alpha\left(\pi - \arctan\left[\frac{\sqrt{1-\beta^2}}{\beta}\right]\right)\right\}}{\sin(\pi\alpha)} \end{aligned} \tag{8}$$

$-1 \leq \beta < 0$ case

In this range of β we will have $z_1 = \beta + i\sqrt{1-\beta^2} \equiv e^{i(\pi-\theta)}$ where $\theta = \arctan\left[\frac{\sqrt{1-\beta^2}}{|\beta|}\right]$, and $z_2 = \beta - i\sqrt{1-\beta^2} \equiv e^{i(\pi+\theta)}$. Note that we again defined the angle of z_2 so that we don't cross the branch cut. z_1 is in the second quadrant and z_2 is in the third. We can write I as

$$\begin{aligned} I &= \frac{\pi i}{i\sqrt{1-\beta^2}(1-e^{i2\pi\alpha})} e^{i\pi\alpha} \left[e^{-i\theta\alpha} - e^{i\theta\alpha}\right] = \frac{\pi}{\sqrt{1-\beta^2}} \frac{\sin\left[\alpha(\theta)\right]}{\sin(\pi\alpha)} \\ &= \frac{\pi}{\sqrt{1-\beta^2}} \frac{\sin\left\{\alpha \arctan\left[\frac{\sqrt{1-\beta^2}}{|\beta|}\right]\right\}}{\sin(\pi\alpha)} \end{aligned} \tag{9}$$

$\beta < -1$ case

In this range of β we will have $z_{1,2} = \beta \pm \sqrt{\beta^2 - 1}$, which are both negative real numbers. We can write I as

$$\begin{aligned} I &= \frac{\pi i}{\sqrt{\beta^2-1}(1-e^{i2\pi\alpha})} e^{i\pi\alpha} \left[\left(|\beta| - \sqrt{\beta^2-1} \right)^\alpha - \left(|\beta| + \sqrt{\beta^2-1} \right)^\alpha \right] \\ &= \frac{\pi \left[\left(|\beta| + \sqrt{\beta^2-1} \right)^\alpha - \left(|\beta| - \sqrt{\beta^2-1} \right)^\alpha \right]}{2\sqrt{\beta^2-1}\sin(\pi\alpha)} \end{aligned} \tag{10}$$

Verifying with Mathematica

The results can be verified with Mathematica. Find the code here.