Integral of the month: $\int dx \frac{\sin^2 x}{x^2}$

2025-09-09

Four different ways of evaluating this lovely integral! We explore contour integration techniques using both upper and lower semicircular contours, as well as a clever approach involving shifted contours to avoid pole splitting. A couple more approaches come from other sneaky techniques.

blog: link

email: quarktetra@gmail.com

We want to compute the integral $I = \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2}$ in various ways.

A complex contour integration

As it is typically done, we first upgrade real valued parameter x to a complex number z, and use the following equality:

$$\frac{\sin^2 z}{z^2} = \frac{\left(e^{iz} - e^{-iz}\right)^2}{-4z^2} = -\frac{e^{2iz} - 2 + e^{-2iz}}{4z^2} = \frac{1 - e^{-2iz}}{4z^2} + \frac{1 - e^{2iz}}{4z^2}. \tag{1}$$

We can evaluate the integrals of the terms on the right hand side using appropriate closed contours. e^{iz} term, for example, requires us to close the contour from above, as in Figure 1, such that the imaginary part of z is positive, which implies $Re(iz) < 0$. As $R \to \infty$, the integral over the large circle vanishes.

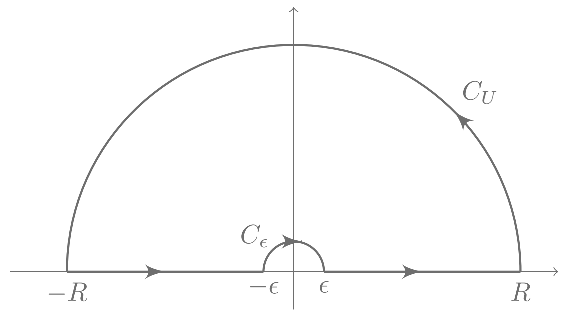

Figure 1: The complex contour in which the singularity at the origin is avoided by bending the curve around it. It is closed from above to make sure exponential term, e^{iz}, vanishes as $R \to \infty$.

Let's evaluate the integral of $\frac{1-e^{2iz}}{z^2}$ over C_U first.

$$
\begin{aligned}
I_U &\equiv \oint_{C_U} dz \frac{1-e^{2iz}}{4z^2} \\
&= \int_{-R}^{\varepsilon} dx \frac{1-e^{2ix}}{4x^2} + \int_{\varepsilon_{U,C}} dz \frac{1-e^{2iz}}{4z^2} + \int_{\varepsilon}^{R} dx \frac{1-e^{2ix}}{4x^2} + \cancel{\int_0^{\pi} Rid\phi e^{i\phi} \frac{1-e^{2iRe^{i\phi}}}{4R^2 e^{2i\phi}}} \\
&= \int_{-R}^{R} dx \frac{1-e^{2ix}}{4x^2} + \int_{\varepsilon_{U,C}} dz \frac{1-e^{2iz}}{4z^2}, \qquad (2)
\end{aligned}
$$

where the integral on the large circle vanishes in the $R \to \infty$. The second term in the last line, $\int_{\varepsilon_{U,C}}$, is to be evaluated on the upper part of the small circle in the clockwise direction, but we can flip its direction by the change of variable $z \to -z$

$$
\int_{\varepsilon_{U,C}} dz \frac{1-e^{-2iz}}{4z^2} = \int_{\varepsilon_{U,CC}} dz \frac{1-e^{2iz}}{4z^2}, \qquad (3)
$$

which will be useful later.

Let's look at the second piece in Eq. 1, and integrate it over the contour Figure 2.

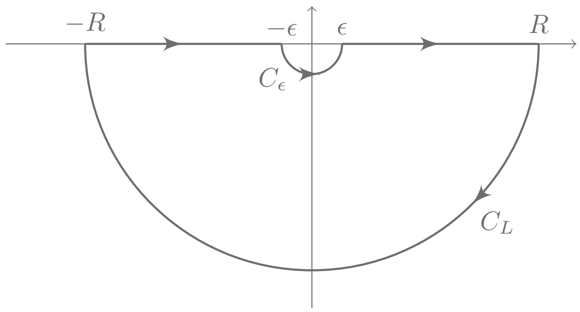

Figure 2: The complex contour in which the singularity at the origin is avoided by bending the curve around it. It is closed from below to make sure exponential term,e^{-iz} , vanishes as $R \to \infty$.

$$
\begin{aligned}
I_L &\equiv \oint_{C_L} dz \frac{1-e^{-2iz}}{4z^2} \\
&= \int_{-R}^{\varepsilon} dx \frac{1-e^{-2ix}}{4x^2} + \int_{\varepsilon_{L,CC}} dz \frac{1-e^{-2iz}}{4z^2} + \int_{\varepsilon}^{R} dx \frac{1-e^{-2ix}}{4x^2} + \cancel{\int_0^{\pi} Rid\phi e^{-i\phi} \frac{1-e^{-2iRe^{i\phi}}}{4R^2 e^{-2i\phi}}} \\
&= \int_{-R}^{R} dx \frac{1-e^{-2ix}}{4x^2} + \int_{\varepsilon_{L,CC}} dz \frac{1-e^{-2iz}}{4z^2}. \qquad (4)
\end{aligned}
$$

Adding Eqs. 2 and 4 we get

$$
\begin{aligned}
I_U + I_L &= \int_{-R}^{R} dx \left(\frac{1-e^{-2ix}}{4x^2} + \frac{1-e^{2ix}}{4x^2} \right) + \left(\int_{\varepsilon_{U,CC}} + \int_{\varepsilon_{L,CC}} \right) dz \frac{1-e^{-2iz}}{4z^2} \\
&= \int_{-R}^{R} dx \frac{\sin^2 x}{x^2} + \oint_{\varepsilon_{CC}} dz \frac{1-e^{-2iz}}{4z^2} = \int_{-R}^{R} dx \frac{\sin^2 x}{x^2} + 2\pi i \left[\frac{d}{dz} \frac{1-e^{-2iz}}{4} \right]_{z=0} \\
&= \int_{-R}^{R} dx \frac{\sin^2 x}{x^2} - \pi. \qquad (5)
\end{aligned}
$$

We know that none of the closed contours we used enclose any poles which means $I_U + I_L = 0$. Therefore:

$$
\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \pi. \qquad (6)
$$

Another complex contour integration

Note that the difficulty we had in the previous section can be traced back to the fact that we split the pole at $z = 0$ so that it was kind of shared between two contours. At the end of the day, the parts came together to give us a closed contour integral. We can get around this by shifting the contours downward as in Figure 3.

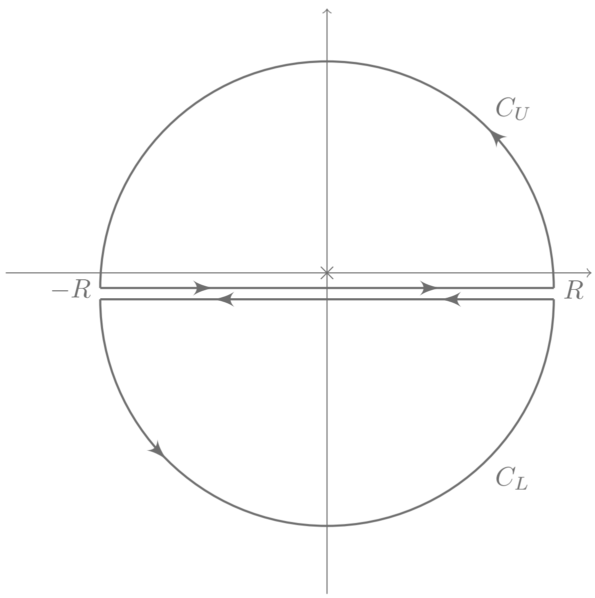

Figure 3: The complex semi-circles shifted downward by a small amount ϵ. One of the contours enclose the pole and the other does not.

Now the top contour includes the pole and the bottom one excludes it. Evaluating and adding the contour integrals we get:

$$\begin{aligned}
I_U &\equiv \oint_{C_U} dz \frac{1-e^{2iz}}{4z^2} = \lim_{R\to\infty} \int_{-R}^{R} dx \frac{1-e^{2ix}}{4x^2} + \cancel{\lim_{R\to\infty} \int_0^{\pi} Rid\phi e^{i\phi} \frac{1-e^{2iRe^{i\phi}}}{4R^2 e^{2i\phi}}} \\
&= 2\pi i \left[\frac{d}{dz} \frac{1-e^{-2iz}}{4} \right] = \pi, \\
I_L &\equiv \oint_{C_U} dz \frac{1-e^{-2iz}}{4z^2} = \lim_{R\to\infty} \int_{-R}^{R} dx \frac{1-e^{-2ix}}{4x^2} + \cancel{\lim_{R\to\infty} \int_0^{\pi} Rid\phi e^{i\phi} \frac{1-e^{2iRe^{i\phi}}}{4R^2 e^{2i\phi}}} \\
&= 0, \\
I_U + I_L &= \lim_{R\to\infty} \left[\int_{-R}^{R} dx \frac{1-e^{2ix}}{4x^2} + \int_{-R}^{R} dx \frac{1-e^{-2ix}}{4x^2} \right] = \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \pi. \qquad (7)
\end{aligned}$$

Fourier transform

If you have ever taken any Signals&Systems Engineering classes, or quantum physics classes, you will remember that Fourier transform of a window function goes like $\frac{\sin w}{w}$ in the frequency domain. Coupling this information with the Parseval's identity, we can solve the integral in the time domain. Let's go through some definitions:

$$\begin{aligned} F(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-iwt} f(t), \\ f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw e^{iwt} F(w). \end{aligned} \tag{8}$$

And the Parseval's identity is easy to prove

$$\begin{aligned} \int_{-\infty}^{\infty} dw F^*(w) F(w) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tilde{t} e^{-iw(t-\tilde{t})} f(t) f^*(\tilde{t}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tilde{t} \left[\int_{-\infty}^{\infty} dw e^{-iw(t-\tilde{t})} \right] f(t) f^*(\tilde{t}), \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tilde{t} 2\pi \delta(t-\tilde{t}) f(t) f^*(\tilde{t}) = \int_{-\infty}^{\infty} dt |f^2(t)|. \end{aligned} \tag{9}$$

If we can find a function, $f(t)$, Fourier transform of which gives $\frac{\sin w}{w}$, we can evaluate $\int_{-\infty}^{\infty} dt |f^2(t)|$ rather than $\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2}$, and Parseval's identity ensures that the results will be the same. One can verify that the follwing function is what we are looking for:

$$f(t) = \sqrt{\frac{\pi}{2}} \left[\theta(t-1) - \theta(t+1) \right], \tag{10}$$

where $\theta(t)$ is the unit step function. $f(t)$ is simply equals to 1 for $-1 < t < 1$ and 0 elsewhere. Using the Parseval's identity we get:

$$\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \int_{-\infty}^{\infty} dt |f^2(t)| = \frac{\pi}{2} \int_{-1}^{1} dt = \pi. \tag{11}$$

A sneaky method

As you fool around with such integrals, you will develop various tricks to generalize them by inserting a parameter inside the integrand. Consider the following object:

$$I(\alpha) = \int_{-\infty}^{\infty} dx \frac{\sin^2 \alpha x}{x^2}, \tag{12}$$

which looks even harder to evaluate. How about $\frac{dI}{d\alpha}$? Let's try:

$$\frac{dI(\alpha)}{d\alpha} = \int_{-\infty}^{\infty} dx \frac{d\sin^2 \alpha x}{d\alpha} x^2 = \int_{-\infty}^{\infty} dx \frac{\sin(2\alpha x)}{x} = \int_{-\infty}^{\infty} dy \frac{\sin(y)}{y}, \tag{13}$$

where $\int_{-\infty}^{\infty} dy \frac{\sin(y)}{y}$ is an easier integral to compute, and its value is π (see this post for various ways to evaluate the integral.)

Since we know $\frac{dI(\alpha)}{d\alpha}$, we can integrate to get

$$I(\alpha) = \int_0^{\alpha} d\tilde{\alpha} \frac{dI(\tilde{\alpha})}{d\tilde{\alpha}} = \alpha\pi + I(0). \tag{14}$$

But we know that the integrand vanishes at $\alpha = 0$, that is $I(0) = 0$, and the integral we are looking for is at $\alpha = 1$. So the result is

$$\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = I(1) = \pi. \tag{15}$$

There you have it, four different ways of evaluating this lovely integral.