Integral of the month: $\int dx \frac{\sin x}{x}$

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Three different ways of evaluating this lovely integral! We explore complex contour integration using the upper semicircular contour to avoid the singularity at the origin. The parametric Laplace transform method introduces a parameter and manipulates it to simplify the integral evaluation. Finally, we demonstrate the direct Laplace transform approach, showing how to create the necessary 1/x term in the integrand.

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We want to compute the integral $I = \int_{-\infty}^{\infty} dx \frac{\sin x}{x}$ in various ways.

A complex contour integration

As it is typically done, we first upgrade real valued parameter x to a complex number z and then construct the contour in Figure 1.

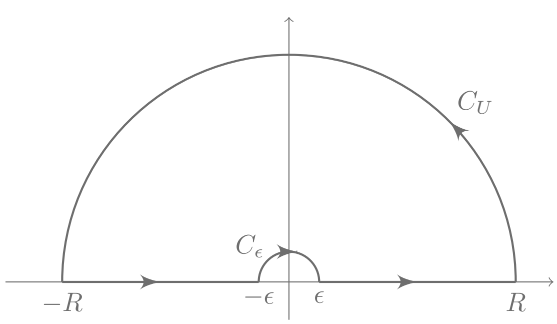

Figure 1: The complex contour in which the singularity at the origin is avoided by bending the curve around it. It is closed from above to make sure exponential term, e^{iz}, vanishes as $R \to \infty$.

On the circle of radius ε, $z = \varepsilon e^{i\theta}$ where $\theta \in [0, \pi]$. And on the large circle $z = Re^{i\phi}$ where $\phi \in [0, \pi]$. We can easily evaluate the following integral (in the limit $\varepsilon \to 0$ and $R \to \infty$):

$$\begin{aligned} I_c &\equiv \oint dz \frac{e^{iz}}{z} = \int_{-R}^{\varepsilon} dx \frac{e^{ix}}{x} + \int_{\pi}^{0} \varepsilon i d\theta e^{i\theta} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} + \int_{\varepsilon}^{R} dx \frac{e^{ix}}{x} + \int_{0}^{\pi} Rid\phi e^{i\phi} \frac{e^{iRe^{i\phi}}}{Re^{i\phi}} \\ &= \int_{-R}^{R} dx \frac{e^{ix}}{x} + \int_{\pi}^{0} id\theta + \int_{0}^{\pi} id\phi e^{iRe^{i\phi}} = \int_{-R}^{R} dx \frac{e^{ix}}{x} - i\pi. \end{aligned} \tag{1}$$

Note that the integral over the large circle vanishes as $R \to \infty$ since $e^{iRe^{i\phi}} = e^{-R\sin\phi} e^{iR\cos\phi}$. Therefore, by explicit evaluation, we see that $I_c = \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} + i\pi$. But, from the theory of residues, we know that the closed loop integral of a function is 0 if the contour does not enclose any poles. Therefore

$$\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi, \tag{2}$$

and if we take the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi. \tag{3}$$

Side note: we evaluated the integral over the inner half circle explicitly. We could also see that it would give $i\pi$ by observing that it is half of a circle that would have enclosed the singularity at the origin. Integral over the full circle would give $2\pi i$, and the integral over the upper-half gives $i\pi$.

Parametric Laplace transform

One of my favorite tricks in integration is to introduce a parameter in the integrand and manipulate it to simplify the integral. Let us insert an α parameter in sin:

$$I(\alpha) = \int_{-\infty}^{\infty} dx \frac{\sin(\alpha x)}{x}. \tag{4}$$

Let us apply a Laplace transform with respect to α to be followed by the inverse Laplace transform

$$\begin{aligned} I &= \mathcal{L}^{-1}\left[\mathcal{L}[I(\alpha)]\right] = \mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} dx \mathcal{L}\left[\frac{\sin(\alpha x)}{x}\right]\right] = \mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} dx \frac{1}{s^2 + x^2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s}\int_{-\infty}^{\infty} d(x/s) \frac{1}{1 + (x/s)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\arctan(x/s)\Big|_{-\infty}^{\infty}\right] = \pi\mathcal{L}^{-1}\left[\frac{1}{s}\right] \\ &= \pi. \end{aligned} \tag{5}$$

Direct Laplace transform

Here is a reminder on the definition of the Laplace transform:

$$F(s) = \mathcal{L}\left[f\right] = \int_0^\infty dx e^{-sx} f(x). \tag{6}$$

From the definition, we can see that we can create a $\frac{1}{x}$ term in the integrand if we simply integrate left side from s to ∞:

$$\int_s^\infty d\tilde{s} F(\tilde{s}) = \int_0^\infty dx \left[\int_s^\infty d\tilde{s} e^{-\tilde{s}x}\right] f(x). = \int_0^\infty dx e^{-sx} \frac{f(x)}{x}. \tag{7}$$

Therefore, if we have an expression of the form $f(x)/x$, we can transform it as $\int_s^\infty d\tilde{s} F(\tilde{s})$. In our case $f(x) = \sin x$ and $F(s) = \frac{1}{1+s^2}$. Using the property above we get

$$I_s = \int_0^\infty dx e^{-sx} \frac{\sin x}{x} = \int_s^\infty d\tilde{s} F(\tilde{s}) = \int_s^\infty d\tilde{s} \frac{1}{1+\tilde{s}^2} = \arctan \tilde{s}\bigg|_s^\infty = \pi/2 - \arctan s. \tag{8}$$

Note that I_s at $s = 0$ is half of the integral we are looking for: $\int_{-\infty}^\infty dx \frac{\sin x}{x} = 2\int_0^\infty dx \frac{\sin x}{x}$. Doubling the result at $s = 0$ yields:

$$I = 2I_0 = \pi - 2\arctan(0) = \pi \tag{9}$$

There you have it, three ways of evaluating this lovely integral.