

L4C Seismometer transfer function

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A seismometer is essentially a mass-spring system with a resonance frequency of 1 Hz, where the motion of the coil contributes to its impedance. We derive the transfer function of the system using Laplace transforms, starting from the differential equation for a damped and forced harmonic oscillator.

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A seismometer, or a geophone, is essentially a mass-spring system with a resonance frequency of 1 Hz. It is a dynamic device, as shown in Figure 1 and the motion of the coil will contribute to its impedance.

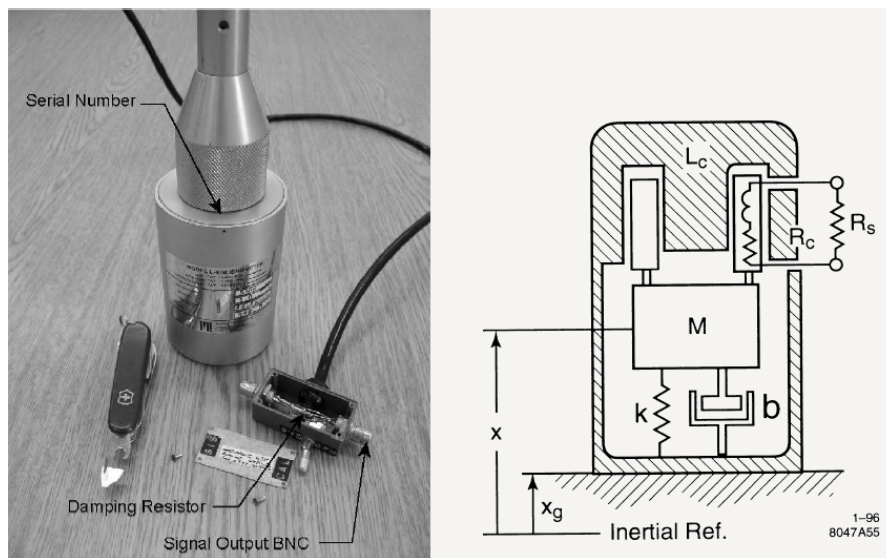


Figure 1: Left: an image of L4C geophone, right: cartoon of inner mechanics. Credit: [1]

The oscillations of the mass are read out by coupling a magnet to a pickup coil. We first want to understand the transfer function associated with this system. There is a one-to-one mapping between the parameters of a mass-spring system and those of an RLC circuit. Figure 2 illustrates this correspondence.

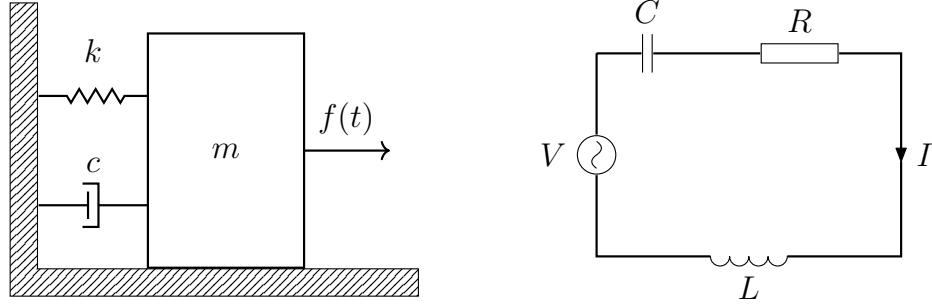


Figure 2: Left: Mass–spring–damper driven by an external force. Right: RLC circuit driven by an AC source.

Table 1: Descriptions of the parameters

Spring-Mass		RLC circuit	
Parameter	Description	Parameter	Description
k	Spring constant	C	Capacitance
c	Damping coefficient	R	Resistance
m	mass of the object	L	Inductance
$f(t)$	External force	$V(t)$	External Voltage

Although they are totally different physical systems, the differential equations governing them are very similar, and they can be written as:

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = f(t) \quad (\text{Newton's second law}) \quad (1)$$

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V(t) \quad (\text{Kirchhoff's Voltage Law}). \quad (2)$$

$$(3)$$

Analytical model

Let's concentrate on Eq. 1, and divide the equation by m . The simplified differential equation for forced harmonic oscillator with damping reads:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{f(t)}{m}, \quad x(0) = x_0, \text{ and } \dot{x}(0) = \dot{x}_0, \quad (4)$$

where $\dot{x} \equiv \frac{dx}{dt}$, $\omega_0 \equiv \sqrt{\frac{k}{m}}$ is the natural frequency of the oscillation, and $\zeta \equiv \frac{c}{2\sqrt{mk}}$ is the *damping ratio*. We also included the initial conditions.

We are dealing with an inhomogeneous linear differential equation with constant coefficients. One of the best tools to solve such equations is the Laplace transformation:

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty dt e^{-st} x(t). \quad (5)$$

The nice feature of the Laplace transformation is that it converts differential equations to algebraic equations. It follows from the transformation property of the derivatives:

$$\begin{aligned} \mathcal{L}[\dot{x}(t)] &= \int_0^\infty dt e^{-st} \frac{dx}{dt} = \int_0^\infty dt \frac{d}{dt} (e^{-st} x) - \int_0^\infty dt \left(\frac{d}{dt} e^{-st} \right) x \\ &= \left(e^{-st} x \right) \Big|_0^\infty + s \int_0^\infty dt e^{-st} x = sX(s) - x_0. \end{aligned} \quad (6)$$

Similarly, the second-order derivative transforms as

$$\mathcal{L}[\ddot{x}(t)] = s\mathcal{L}[\dot{x}(t)] - \dot{x}_0 = s^2X(s) - sx_0 - \dot{x}_0. \quad (7)$$

Laplace transforming Eq. 4 we get

$$s^2X - sx_0 - \dot{x}_0 + 2\zeta\omega_0(sX - x_0) + \omega_0^2X = \frac{1}{m}F(s). \quad (8)$$

Solving Eq. 8 for X , we get

$$\begin{aligned}
X &= \frac{sx_0 + 2\zeta\omega_0x_0 + \dot{x}_0}{s^2 + 2\zeta\omega_0s + \omega_0^2} + \frac{1}{m} \frac{F(s)}{s^2 + 2\zeta\omega_0s + \omega_0^2} \\
&= \frac{(s + \zeta\omega_0)x_0 + \zeta\omega_0x_0 + \dot{x}_0}{(s + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2)} + \frac{1}{m} \frac{F(s)}{(s + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2)}.
\end{aligned} \tag{9}$$

The first term in Eq. 9 is related to the initial conditions, and the excitations associated with this term will die out due to damping. The second term relates the response of the system to the external force. The transfer function of the system is given by

$$H(s) = \left. \frac{X(s)}{F(s)} \right|_{x_0=0=\dot{x}_0} = \frac{1}{m} \frac{1}{(s + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2)}. \tag{10}$$

Response to a sinusoidal force

Although we will concentrate mostly on the transfer function, it is possible to evaluate the inverse Laplace transform if the functional form of the driving force is known. It is a good exercise to calculate the time domain functions when the system is driven by a sinusoidal force. Let's assume that $f(t)$ is of the following form:

$$f(t) = f_0 \sin(\omega t). \tag{11}$$

Its Laplace transform is given by:

$$F(s) \equiv \mathcal{L}[f(t)] = \frac{f_0\omega}{s^2 + \omega^2}. \tag{12}$$

We will have to do some partial fraction expansion:

$$\frac{1}{((s + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2))(s^2 + \omega^2)} = \frac{A(s + \zeta\omega_0) + B}{(s + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2)} + \frac{Cs + D}{s^2 + \omega^2}, \tag{13}$$

which will be easy to convert back to time domain since they will correspond to sines and cosines with exponential functions in front. We now need to figure out A , B , C and D . If we were to equate the denominators and sum up the resulting numerators, we will see that, in order to set the coefficient of the s^3 term in the numerator to zero we will need $A = -C$. To

relate C and D we can multiply 13 by $s - i\omega$ and then set $s = i\omega$. This will remove the first term on the right-hand side and yield:

$$i\omega C + D = \frac{1}{(i\omega + \zeta\omega_0)^2 + \omega_0^2(1 - \zeta^2)}. \quad (14)$$

This is a complex equation, and splitting it into the real and imaginary part, we get:

$$\begin{aligned} D(\omega_0^2 - \omega^2) - 2C\omega^2\omega_0\zeta &= 1, \\ C\omega(\omega_0^2 - \omega^2) + 2D\omega\omega_0\zeta &= 0. \end{aligned} \quad (15)$$

Inverting it results in:

$$\begin{aligned} C &= \frac{-2\omega_0\zeta}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2}, \\ D &= \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2}. \end{aligned} \quad (16)$$

Finally, setting $s = -\zeta\omega_0$, and going through some algebra, we get:

$$B = \frac{\omega^2 - \omega_0^2 + 2\omega_0^2\zeta^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2}. \quad (17)$$

We can now inverse transform Eq. 9 using elementary properties of the transformation:

$$x(t) = \mathcal{L}^{-1}[X(s)]. \quad (18)$$

Inverse Laplace transformation yields:

$$\begin{aligned} x(t) &= \left[x_0 \cos(\omega_0 \sqrt{1 - \zeta^2} t) + \frac{\zeta\omega_0 x_0 + \dot{x}_0}{\omega_0 \sqrt{1 - \zeta^2}} \sin(\omega_0 \sqrt{1 - \zeta^2} t) \right] e^{-\zeta\omega_0 t} \\ &+ \frac{2f_0\omega}{m} \frac{\omega_0\zeta}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2} e^{-\zeta\omega_0 t} \cos(\omega_0 \sqrt{1 - \zeta^2} t) \\ &+ \frac{f_0\omega}{m} \frac{1}{\omega_0 \sqrt{1 - \zeta^2}} \frac{\omega^2 - \omega_0^2 + 2\omega_0^2\zeta^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2} e^{-\zeta\omega_0 t} \sin(\omega_0 \sqrt{1 - \zeta^2} t) \\ &- \frac{2f_0\omega}{m} \frac{\omega_0\zeta}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2} \cos(\omega t) + \frac{f_0}{m} \frac{\omega^2 - \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\omega_0^2\zeta^2} \sin(\omega t). \end{aligned} \quad (19)$$

We can do one last touch and combine the last two terms into a single function with a phase shift. The full solution with damping, and with $f(t) = f_0 \sin(\omega t)$, can be written as:

$$\begin{aligned}
x(t) = & \left[x_0 \cos(\omega_0 \sqrt{1 - \zeta^2} t) + \frac{\zeta \omega_0 x_0 + \dot{x}_0}{\omega_0 \sqrt{1 - \zeta^2}} \sin(\omega_0 \sqrt{1 - \zeta^2} t) \right] e^{-\zeta \omega_0 t} \\
& + \frac{f_0 \omega e^{-\zeta \omega_0 t}}{m[(\omega_0^2 - \omega^2)^2 + 4\omega^2 \omega_0^2 \zeta^2]} \left[2\omega_0 \zeta \cos(\omega_0 \sqrt{1 - \zeta^2} t) + \frac{\omega^2 - \omega_0^2 + 2\omega_0^2 \zeta^2}{\omega_0 \sqrt{1 - \zeta^2}} \sin(\omega_0 \sqrt{1 - \zeta^2} t) \right] \\
& + \frac{f_0}{m \sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \omega_0^2 \zeta^2}} \sin(\omega t - \delta), \tag{20}
\end{aligned}$$

where $\delta \equiv \arctan \left[\frac{2\omega \omega_0 \zeta}{\omega_0^2 - \omega^2} \right]$, the first line is related to the initial conditions, the second and third lines are the transient response, and finally the last line is the steady state solution.

- At later times, $t \gg 1/(\zeta \omega_0)$, i.e., in the steady state, only the last term survives.
 - The $x(t)$ is sinusoidal, but it will lag by a phase δ .
- The system will enter resonance at $\omega = \omega_0 \sqrt{1 - 2\zeta^2}$.
 - The value of the resonance amplitude is $f_0/(2\omega_0^2 \zeta \sqrt{1 - \zeta^2})$.
 - At $\zeta = 0$ (no damping), the amplitude diverges. We need to go back and study this case carefully.

Resonances at zero damping:

The final solution runs into problems when we consider $\zeta = 0$ and $\omega = \omega_0$: the coefficient of the steady state solution diverges. This is because of the assumptions we made when we were inverting $X(s)$. At $\zeta = 0$ and $\omega = \omega_0$, two poles will merge and create a second order pole. Let's take a closer look:

$$\lim_{\zeta \rightarrow 0, \omega \rightarrow \omega_0} \frac{1}{((s + \zeta \omega_0)^2 + \omega_0^2(1 - \zeta^2))(s^2 + \omega_0^2)} = \frac{1}{(s^2 + \omega_0^2)^2}. \tag{21}$$

We can figure out how to inverse transform it by exploiting a few features of the Laplace transforms as follows:

$$\begin{aligned}
\mathcal{L}^{-1} \left[\frac{1}{(s^2 + \omega_0^2)^2} \right] &= \mathcal{L}^{-1} \left[-\frac{1}{2s} \frac{d}{ds} \left(\frac{1}{s^2 + \omega_0^2} \right) \right] = -\frac{1}{2} \int_0^t d\tau \tau \frac{\sin(\omega_0 \tau)}{\omega_0} \\
&= -\frac{1}{2\omega_0} \frac{d}{d\omega_0} \int_0^t d\tau \cos(\omega_0 \tau) \\
&= -\frac{1}{\omega_0} \frac{d}{d\omega_0} \left[\frac{\sin(\omega_0 t)}{\omega_0} \right] = \frac{\sin(\omega_0 t) - \omega_0 t \cos(\omega_0 t)}{2\omega_0^3}.
\end{aligned} \tag{22}$$

The full solution at the resonance frequency ($\omega = \omega_0$) with no damping ($\zeta = 0$) is:

$$x(t) = \left[x_0 \cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t) \right] + \left[\frac{f_0}{2m\omega_0^2} (\sin(\omega_0 t) - \omega_0 t \cos(\omega_0 t)) \right]. \tag{23}$$

This shows that the amplitude will grow with time. In reality the model will break at some point since the amplitude of oscillations cannot grow indefinitely. (For example, the spring will literally break if it is stretched too far.)

Impedance vs frequency

In order to compute the effective impedance of the sensor, we can poke it with a test current $i(t)$ and compute the corresponding voltage across the device. Let us start from Eq. 4 and replace the right-hand side with the force created by the probing current:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = \frac{Bli}{m}, \tag{24}$$

where B is the magnetic field of the permanent magnet attached to the test mass and l is the effective length of the coil. The probing current creates a force on the permanent magnet and the test mass moves to create an induced voltage. When operated around the natural frequency of the device, the voltage is proportional to the velocity:

$$v_g(t) = Bl\dot{x}. \tag{25}$$

Note that we are trying to isolate the impedance associated with the dynamics. The seismometer has its intrinsic resistance (R) and inductance (L) which we will simply add on as $Ls + R$.

We can easily convert the relation between x and i in Eq. 24 into a relation between v_g and i by taking another derivative:

$$\frac{d^2}{dt^2}\dot{x} + 2\zeta\omega_0\frac{d}{dt}\dot{x} + \omega_0^2\dot{x} = \frac{Bl\dot{i}}{m}. \quad (26)$$

Replacing \dot{x} with $\frac{v_g}{Bl}$ yields

$$\ddot{v}_g + 2\zeta\omega_0\dot{v}_g + \omega_0^2v_g = \frac{B^2l^2\dot{i}}{m}. \quad (27)$$

In the s space, the relation between the current and the voltage is given by:

$$V_g(s) = \frac{B^2l^2}{m} \frac{s}{s^2 + 2\zeta\omega_0s + \omega_0^2} I(s) \equiv Z(s)I(s), \quad (28)$$

where we identify the factor in front of $I(s)$ as the impedance associated with the dynamics of the mass. Finally, we can add the intrinsic impedance of the coil to get the complete impedance as

$$\begin{aligned} Z(s) &= Ls + R + G^2 \frac{s}{s^2 + 2\zeta\omega_0s + \omega_0^2} \\ &= \frac{Ls^3 + (2L\zeta\omega_0 + R)s^2 + (2\zeta\omega_0R + \omega_0^2L + G^2)s + R\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}, \end{aligned} \quad (29)$$

where we defined $G^2 = \frac{B^2l^2}{m}$.

Table 2 shows the values of the device parameters.

We can plot the impedance as a function of frequency by plugging in $s = j2\pi f$ and the numerical values from Table 2 into Eq. 29. The impedance can be computed as follows:

$$Z(f) = \left. \frac{6.05s^3 + 5521s^2 + 95976s + 217131}{s^2 + 3.52s + 39.48} \right|_{s=j2\pi f}. \quad (30)$$

Figure 3 shows the impedance of the device.

Table 2

[!h]

Table 3: Descriptions of the parameters

Parameter	Description	Value
m	mass	1[kg]
R	Resistance	5500[Ω]
L	Inductance	6050[mH]
G	Coupling	276[$\frac{V}{m/s}$]
ζ	Damping	0.28

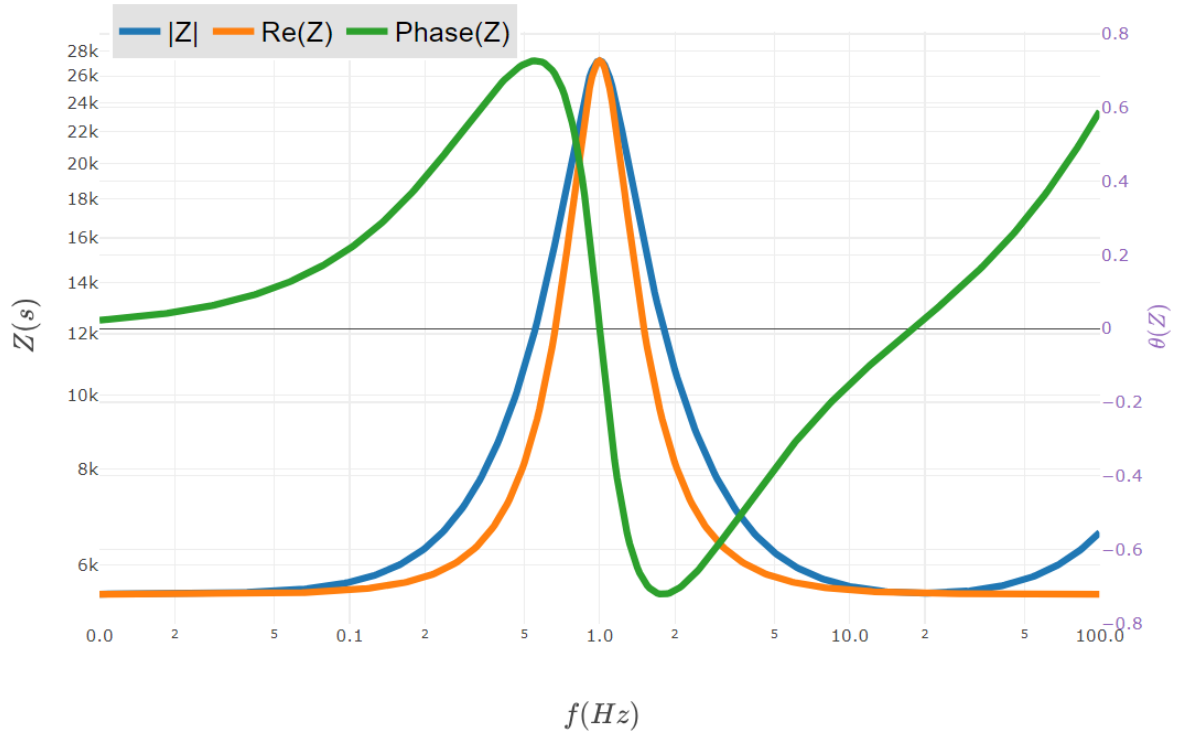


Figure 3: Impedance as a function of the frequency. Right axis shows the angle of the impedance.

- [1] G. B. Bowden, “Calibration of geophone microseismic sensors.” 2003.