Magnetic Dipole

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This blog post provides a comprehensive treatment of magnetic dipoles, starting from first principles. We begin by deriving the vector potential for an arbitrary current distribution and apply it to the specific case of a circular current loop. The exact solution is expressed in terms of complete elliptic integrals, and we provide explicit forms for the magnetic field in both spherical and cylindrical coordinates. We then develop the magnetic dipole approximation, showing how it emerges naturally as the leading term in a multipole expansion. Finally, we extend our analysis to continuous distributions of magnetic moments, introducing the concept of bound currents and demonstrating how they provide an elegant framework for describing magnetized materials. Throughout, we emphasize the mathematical techniques and physical insights that connect these various aspects of magnetic dipole physics.

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Vector Potential

The magnetic field at an arbitrary point $\mathbf{r}$ created by a current distribution $\mathbf{J}(\mathbf{r}')$ is given by the Biot-Savart law[1]:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\,\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^3}, \tag{1}$$

where we use the primed coordinates for the source points. We can convert this to a curl of a vector potential using the following identity:

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{2}$$

Putting this into Eq. 1 we get:

$$\mathbf{B}(\mathbf{r}) \quad = \quad -\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\mathbf{J}(\mathbf{r}')\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \nabla\times\left[\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] \equiv \nabla\times\mathbf{A}(\mathbf{r}). \tag{3}$$

Equation 3 enables us to define a vector potential for an arbitrary current distribution:

$$\mathbf{A}(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4}$$

Vector Potential of a Single Loop

We consider the magnetic field of a single circular loop with a current as in Figure 1.

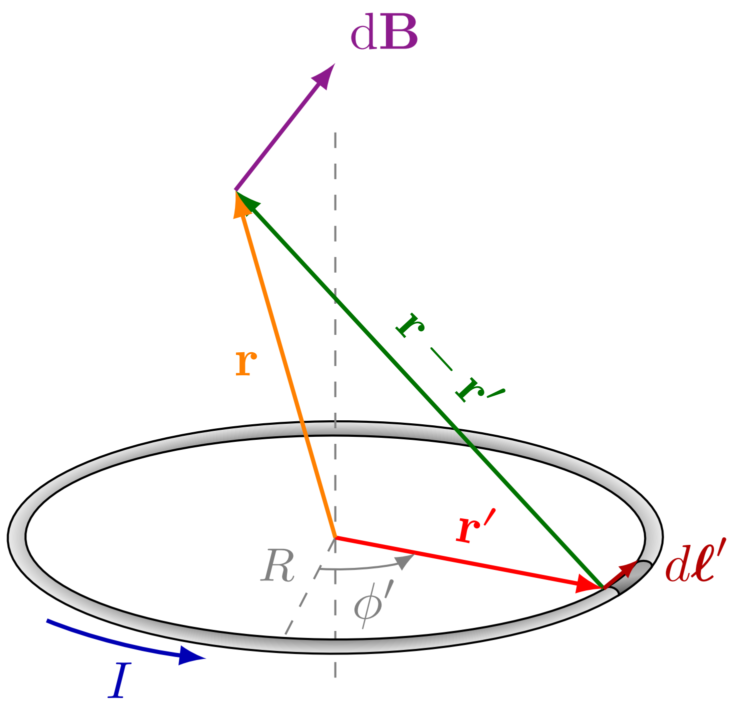

Figure 1: A loop of wire with radius R carrying a current I.

For a single loop sitting at $z = 0$ with radius $r = R$, it is convenient to work in spherical coordinates.

$$\mathbf{J}(\mathbf{r}') \quad = \quad \lambda\delta(r'-R)\delta(\cos\theta')\,\hat{\phi}' = \lambda\delta(r'-R)\delta(\cos\theta')\left(\cos\phi'\hat{\mathbf{j}} - \sin\phi'\hat{\mathbf{i}}\right), \tag{5}$$

where λ is the current density. In order to calculate λ for a loop of wire carrying a current I, let's intercept the loop with an area perpendicular to it. We can do that by selecting an area on, say, positive x axis (i.e., $\phi' = 0$), pointing along the y axis, i.e., $d\mathbf{S}' = dS'\hat{\mathbf{j}} = r'dr'd\theta'\hat{\mathbf{j}}$. Integrating the current density on this area we should get the total current:

$$\begin{aligned}\int_S d\mathbf{S}' \cdot \mathbf{J}(\mathbf{r}) &= \int_S r'dr'd\theta'\lambda\delta(r'-R)\delta(\cos\theta')\hat{\mathbf{j}} \cdot \left(\cos\phi'\hat{\mathbf{j}} - \sin\phi'\hat{\mathbf{i}}\right)\Big|_{\phi'=0} = R\lambda = I \\ &\implies \lambda = I/R. \end{aligned} \tag{6}$$

Therefore, the properly normalized current is

$$\mathbf{J}(\mathbf{r}') = \frac{I}{R}\delta(r'-R)\delta(z')\,\hat{}\,' = \frac{I}{R}\delta(r'-R)\delta(\cos\theta')\left(\cos\phi'\hat{\mathbf{j}} - \sin\phi'\hat{\mathbf{i}}\right). \tag{7}$$

The integral we have to deal with for a single loop is this:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi R}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(r'-R)\delta(\cos\theta')\left(\cos\phi'\hat{\mathbf{j}} - \sin\phi'\hat{\mathbf{i}}\right), \tag{8}$$

where we put the subscript s to remind us that this is for a single loop. We will parameterize the points on the loop centered at $z = 0$ as $\mathbf{r}' = r'(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$, and the observation point as $\mathbf{r} = r\cos\theta\hat{\mathbf{z}} + r\sin\theta(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}})$

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{r^2\cos^2\theta + (r\sin\theta\cos\phi - r'\cos\phi')^2 + (r\sin\theta\sin\phi - r'\sin\phi')^2} \\ &= \sqrt{r^2 + r'^2 - 2rr'\sin\theta\cos(\phi'-\phi)}. \end{aligned} \tag{9}$$

Note that the problem has rotational symmetry. We can rotate our coordinate system such that the observation point sits on $y = 0$, i.e., $\phi = 0$. Once we are done with the computations, we can rotate the vectors back to general $\mathbf{r}$ point. So let's set $\phi = 0$ in Eq. 9 and rewrite Eq. 8 :

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi R}\int \sin\theta' r'^2 dr' d\phi' \frac{\delta(r'-R)\delta(\cos\theta')\left(\cos\phi'\hat{\mathbf{j}} - \sin\phi'\hat{\mathbf{i}}\right)}{\sqrt{r^2 + r'^2 - 2rr'\sin\theta\cos(\phi'-\phi)}} \\ &= \frac{\mu_0 IR}{4\pi}\left[\int_0^{2\pi} d\phi' \frac{\cos\phi'\hat{\mathbf{j}}}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos(\phi'-\phi)}} \right. \\ &\qquad \left. \cancelto{0}{-\int_0^{2\pi} d\phi' \frac{\sin\phi'\hat{\mathbf{i}}}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos(\phi'-\phi)}}}\right], \end{aligned} \tag{10}$$

where the second term vanishes since the integrand is odd and the integral is evaluated over the full range. Note that we evaluated the integral at $\phi = 0$, and the resulting potential points in $\hat{\mathbf{j}}$ direction. For generic ϕ we can simply rotate the coordinate system about the z axis by ϕ. In this rotated coordinate system $\hat{\mathbf{j}} \to \hat{\phi}$. Therefore the vector potential reads:

$$\mathbf{A}(\mathbf{r}) \;=\; \hat{\phi}\frac{\mu_0 IR}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{r^2+R^2-2rR\sin\theta\cos\phi'}}. \tag{11}$$

Let's define $\phi' = \pi - \phi'$ to get $\cos\phi' = -\cos\phi'$ and rewrite Eq. 11 as:

$$\mathbf{A}(\mathbf{r}) \;=\; -\hat{\phi}\frac{\mu_0 IR}{4\pi}\int_{-\pi}^{\pi} d\phi' \frac{\cos\phi'}{\sqrt{r^2+R^2+2rR\sin\theta\cos\phi'}}. \tag{12}$$

Let's also use the half angle formula: $\cos\phi' = 1 - 2\sin^2\frac{\phi'}{2}$ and reorganize the integral:

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) \;&=\; -\hat{\phi}\frac{\mu_0 IR}{4\pi}\frac{1}{\sqrt{r^2+R^2+2rR\sin\theta}}\int_{-\pi}^{\pi} d\phi' \frac{1-2\sin^2\frac{\phi'}{2}}{\sqrt{1-\frac{4rR\sin\theta}{r^2+R^2+2rR\sin\theta}\sin^2\frac{\phi'}{2}}} \\
&\equiv\; -\hat{\phi}\frac{\mu_0 IR}{4\pi}\frac{1}{\sqrt{r^2+R^2+2rR\sin\theta}}\int_{-\pi}^{\pi} d\phi' \frac{1-2\sin^2\frac{\phi'}{2}}{\sqrt{1-k^2\sin^2\frac{\phi'}{2}}} \\
&=\; -\hat{\phi}\frac{\mu_0 IR}{4\pi}\frac{1}{\sqrt{r^2+R^2+2rR\sin\theta}}\int_{-\pi}^{\pi} d\phi' \left[\frac{1}{\sqrt{1-k^2\sin^2\frac{\phi'}{2}}} - 2\frac{\sin^2\frac{\phi'}{2}}{\sqrt{1-k^2\sin^2\frac{\phi'}{2}}}\right] \\
&=\; -\hat{\phi}\frac{\mu_0 IR}{4\pi k^2}\frac{1}{\sqrt{r^2+R^2+2rR\sin\theta}}\int_{-\pi}^{\pi} d\phi' \left[\frac{k^2-2}{\sqrt{1-k^2\sin^2\frac{\phi'}{2}}} + 2\sqrt{1-k^2\sin^2\frac{\phi'}{2}}\right],
\end{aligned} \tag{13}$$

where $k^2 = \frac{4rR\sin\theta}{r^2+R^2+2rR\sin\theta}$. Finally, we define $\zeta' = \phi'/2$ and split the integration into two pieces to pick an overall factor of 4 to get:

$$\mathbf{A}(\mathbf{r}) = \hat{\phi}\frac{\mu_0 IR}{\pi\sqrt{r^2+R^2+2rR\sin\theta}}\frac{(2-k^2)K(k^2)-2E(k^2)}{k^2}, \tag{14}$$

where the elliptic integral are defined as follows:

$$\begin{aligned}
K(k^2) \;&=\; \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \\
E(k^2) \;&=\; \int_0^{\frac{\pi}{2}} d\theta\sqrt{1-k^2\sin^2\theta}.
\end{aligned} \tag{15}$$

Magnetic Field of a Single Loop

The calculation of the magnetic field is straightforward[2]:

$$
\begin{aligned}
B_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = \frac{\mu_0 I R^2\cos\theta E(k^2)}{\pi\sqrt{r^2+R^2+2rR\sin\theta}(r^2+R^2-2rR\sin\theta)}, \\
B_\theta &= -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) = \frac{\mu_0 I\left[(r^2+R^2\cos(2\theta))E(k^2)-(r^2+R^2-2rR\sin\theta)K(k^2)\right]}{2\pi\sqrt{r^2+R^2+2rR\sin\theta}(r^2+R^2-2rR\sin\theta)\sin\theta}. \quad (16)
\end{aligned}
$$

We can also express the magnetic field in the cylindrical coordinates [2]:

$$
\begin{aligned}
B_\rho &= \frac{\mu_0 I z\left[(R^2+\rho^2+z^2)E(k^2)-(R^2+\rho^2+z^2-2R\rho))K(k^2)\right]}{2\pi\sqrt{R^2+\rho^2+z^2+2R\rho}(R^2+\rho^2+z^2-2R\rho)\rho}, \\
B_z &= \frac{\mu_0 I\left[(R^2-\rho^2-z^2)E(k^2)+(R^2+\rho^2+z^2-2R\rho))K(k^2)\right]}{2\pi\sqrt{R^2+\rho^2+z^2+2R}(R^2+\rho^2+z^2-2R\rho)\rho}. \quad (17)
\end{aligned}
$$

Dipole Approximation

Although we worked out the complete solution for a single loop, it is worth finding an accurate approximation. We can use the multipole expansion and decide on the number of terms to keep. The full multipole expansion is involves the expansion of the potential in terms of the Legendre polynomials. Just to keep things simple, we will use the dipole approximation to get the leading term.

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \hat{\phi}\frac{\mu_0 IR}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{r^2+R^2-2rR\sin\theta\cos\phi'}} \\
&= \hat{\phi}\frac{\mu_0 IR}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{r}\left[1-\frac{R}{r}\sin\theta\cos\phi'\right] \\
&= \hat{\phi}\frac{\mu_0 I\pi R^2}{4\pi r^2}\sin\theta \equiv \hat{\phi}\frac{\mu_0 M}{4\pi r^2}\sin\theta, \quad (18)
\end{aligned}
$$

where we defined the magnetic moment $M = I\pi R^2$. Note that we can absorb $\hat{\phi}$ and $\sin\theta$ into a cross product of the magnetic moment and the unit vector pointing to the observation point:

$$
\mathbf{A}(\mathbf{r}) = \hat{\phi}\frac{\mu_0 M}{4\pi r^2}\sin\theta \equiv \frac{\mu_0 M}{4\pi r^2}\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \frac{\mu_0}{4\pi r^2}\mathbf{M}\times\hat{\mathbf{r}}, \quad (19)
$$

where $\mathbf{M} = I\pi R^2\hat{\mathbf{z}}$ is the magnetic moment in its vector form. It is important to note the striking similarity between the magnetic dipole and the electric dipole, which has the electric potential as follows:

$$V(\mathbf{r}) \quad = \quad \frac{1}{4\pi\epsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}}, \tag{20}$$

where $\mathbf{P} = e\mathbf{d}$ is the polarization vector. The magnetic field corresponding to the potential in Eq. 19 is:

$$\mathbf{B}(\mathbf{r}) \quad = \quad \nabla\times\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{}\right] = \frac{\mu_0}{4\pi r^5}\left[3(\mathbf{M}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{M}\right]. \tag{21}$$

Bound Currents

Let's start from Eq. 19 with the vector potential for created by sources $\mathbf{M}(\mathbf{r}')$:

$$\mathbf{A} \quad = \quad \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}'). \tag{22}$$

We will repeat the trick we used in the previous section see Eq. 2, but this time in the source coordinates:

$$\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \quad = \quad \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{23}$$

We will use this relation to rewrite the potential in Eq. 22 as follows:

$$\mathbf{A} \quad = \quad \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{24}$$

We will want to move the gradient to the other side of the cross product, so we will use the following vector identity:

$$\nabla'\times\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right) \quad = \quad \frac{\nabla'\times\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|} - \mathbf{M}\times\left(\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right). \tag{25}$$

This will result in a vector potential with two components:

$$\mathbf{A} \quad = \quad \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right). \tag{26}$$

The first term is nice and simple, but the second term needs some work. It is in a rather unusual form. Let's us imagine projecting the resulting vector onto a constant vector $\mathbf{c}$:

$$\begin{aligned}\mathbf{c}\cdot\left[\nabla'\times\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right)\right] \quad &= \quad \left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot(\nabla'\times\mathbf{c}) - \nabla'\cdot\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{c}\right) \\ &= \quad -\nabla'\cdot\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{c}\right), \end{aligned} \tag{27}$$

which is in a form that can leverage the Gauss' divergence theorem. The projected form of the second term is:

$$
\begin{aligned}
-\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\mathbf{c}\cdot\left[\nabla'\times\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right)\right] &= \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\nabla'\cdot\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{c}\right) \\
&= \frac{\mu_0}{4\pi}\int d^2\mathbf{S}\cdot\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times\mathbf{c}\right) \\
&= \frac{\mu_0}{4\pi}\int \mathbf{c}\cdot\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times d^2\mathbf{S}\right). \qquad (28)
\end{aligned}
$$

Since $\mathbf{c}$ is arbitrary, the relation is valid for any component. Putting all off it together, we get:

$$
\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{K}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{a}'. \qquad (29)
$$

where we defined the bound current densities as follows:

$$
\mathbf{J}_b \equiv \nabla'\times\mathbf{M}, \text{ and } \quad \mathbf{K}_b \equiv \mathbf{M}\times\hat{\mathbf{n}}, \qquad (30)
$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface.

[1] D. J. Griffiths, *Introduction to electrodynamics*. Pearson, 2013.
[2] J. C. Simpson, J. E. Lane, C. D. Immer, R. C. Youngquist, and T. Steinrock, "Simple analytic expressions for the magnetic field of a circular current loop," 2001 [Online]. Available: https://api.semanticscholar.org/CorpusID:30815892