An Introduction to Magnetohydrodynamics

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A condensed introduction to ideal magnetohydrodynamics (MHD), drawn from Nick Murphy's lecture notes [1]. We derive the continuity equation, decompose the Lorentz force into magnetic tension and pressure, and assemble the remaining ideal MHD closure: adiabatic energy, Faraday's law, ideal Ohm's law, the induction equation, and the divergence constraint. All equations are in cgs units.

[1] TRUE

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Introduction

Ideal magnetohydrodynamics (MHD) couples Maxwell's equations with hydrodynamics to describe the macroscopic behavior of highly conducting fluids such as plasmas. This note condenses Nick Murphy's lecture *Ideal Magnetohydrodynamics* [1], keeping the derivations of the continuity equation, the Lorentz force, and the remaining ideal MHD closure relations. We use **cgs units** throughout.

The continuity equation

Pick a closed volume $\mathcal{V}$ bounded by a fixed surface S containing plasma with mass density ρ. The total mass is

$$M = \int_{\mathcal{V}} \rho \, d\mathcal{V}, \tag{1}$$

and its time derivative is

$$\frac{dM}{dt} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V}. \tag{2}$$

The mass flowing through a surface element $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ (outward normal $\hat{\mathbf{n}}$) is $\rho\mathbf{V} \cdot d\mathbf{S}$. Conservation requires the change in mass inside $\mathcal{V}$ to equal minus the net flux through S:

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} = -\oint_S \rho\mathbf{V} \cdot d\mathbf{S}. \tag{3}$$

Gauss's theorem gives

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V})\right] d\mathcal{V} = 0, \tag{4}$$

which must hold for all volumes, so the integrand vanishes. The **continuity equation in conservative form** is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) = 0. \tag{5}$$

The mass flux is $\rho\mathbf{V}$; source and sink terms (e.g., ionization in a partially ionized plasma) would appear on the right-hand side. Using vector identities, we may rewrite 5 as

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla\rho = -\rho\, \nabla \cdot \mathbf{V}, \tag{6}$$

where the first term on the left is the local time rate and $\mathbf{V} \cdot \nabla\rho$ is the **advective** (directional) derivative along the flow. The compression term $\nabla \cdot \mathbf{V}$ measures divergence of the velocity field: $\nabla \cdot \mathbf{V} < 0$ for converging (compressing) flow, $\nabla \cdot \mathbf{V} > 0$ for diverging flow, and $\nabla \cdot \mathbf{V} \equiv 0$ for an incompressible plasma.

The **Eulerian** form 6 follows the density at a fixed point in space. The equivalent **Lagrangian** form follows a fluid element co-moving with velocity $\mathbf{V}$:

$$\frac{d\rho}{dt} + \rho\, \nabla \cdot \mathbf{V} = 0, \tag{7}$$

where the **material derivative** is

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla. \tag{8}$$

The Lorentz force

The Lorentz force on a single charged particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}\right). \tag{9}$$

Summing over all species gives the current density

$$\mathbf{J} = \sum_{\alpha} n_\alpha q_\alpha \mathbf{V}_\alpha, \tag{10}$$

where α runs over ions and electrons. For a quasineutral plasma with singly charged ions, $n = n_e = n_i$ and

$$\mathbf{J} = en(\mathbf{V}_i - \mathbf{V}_e). \tag{11}$$

Using Ampère's law (introduced below) and vector identities, the Lorentz force per unit volume decomposes into magnetic tension and magnetic pressure terms:

$$\frac{\mathbf{J}\times\mathbf{B}}{c} = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} = \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi} - \nabla\frac{B^2}{8\pi}. \tag{12}$$

Index-form derivation using Levi-Civita

We willuse the **Einstein summation convention**: repeated indices are summed over (e.g. $a_i b_i \equiv \sum_i a_i b_i$). We will use my favorite machinery in vector calculus: the Levi-Civita symbol. In index notation the cross product and curl are

$$(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk}\, a_j b_k, \qquad (\nabla\times\mathbf{B})_i = \varepsilon_{ijk}\,\partial_j B_k, \tag{13}$$

where ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = +1$, even permutations $+1$, odd -1, repeated index 0). Ampère's law is $J_i = (c/4\pi)\,\varepsilon_{ijk}\,\partial_j B_k$, so

$$\left(\frac{\mathbf{J}\times\mathbf{B}}{c}\right)_i = \frac{1}{4\pi}\,\varepsilon_{ijk}\,(\nabla\times\mathbf{B})_j\,B_k = \frac{1}{4\pi}\,\varepsilon_{ijk}\,\varepsilon_{jlm}\,(\partial_l B_m)\,B_k, \tag{14}$$

with the repeated index j summed. Anticommutativity of the cross product gives $\varepsilon_{ijk}(\nabla\times\mathbf{B})_j B_k = -\varepsilon_{ijk} B_j (\nabla\times\mathbf{B})_k$. Using $(\nabla\times\mathbf{B})_k = \varepsilon_{klm}\partial_l B_m$, the second form is $\varepsilon_{ijk}\varepsilon_{klm} B_j\,\partial_l B_m$. The Levi-Civita contraction is

$$\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}, \tag{15}$$

where δ_{ij} is the Kronecker delta. Substituting into $\varepsilon_{ijk}\varepsilon_{klm} B_j\,\partial_l B_m$ and using the Kronecker deltas ($l = i$, $j = m$ in the first term; $m = i$, $j = l$ in the second) gives

$$\varepsilon_{ijk}\,\varepsilon_{klm}\,B_j\,\partial_l B_m = B_m\,\partial_i B_m - B_l\,\partial_l B_i = \tfrac{1}{2}\partial_i B^2 - (B_j\partial_j)B_i. \tag{16}$$

This is the ith component of $\mathbf{B}\times(\nabla\times\mathbf{B}) = \nabla(B^2/2) - (\mathbf{B}\cdot\nabla)\mathbf{B}$. Therefore $(\nabla\times\mathbf{B})\times\mathbf{B} = -\mathbf{B}\times(\nabla\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(B^2/2)$, and

$$\left(\frac{\mathbf{J}\times\mathbf{B}}{c}\right)_i = \frac{1}{4\pi}\left[(B_j\partial_j)B_i - \tfrac{1}{2}\partial_i B^2\right], \tag{17}$$

which is 12.

While $\mathbf{J}\times\mathbf{B}$ is orthogonal to $\mathbf{B}$, each term on the right may have components along $\mathbf{B}$. Define the unit vector along the field, $\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$, and the **curvature vector**

$$\kappa \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = -\frac{\mathbf{R}}{R^2}, \tag{18}$$

where $\mathbf{R}$ points from the center of curvature to the field line and $|\kappa| = R^{-1}$, with $\kappa \cdot \hat{\mathbf{b}} = 0$. With the product rule, $\mathbf{B} \cdot \nabla \mathbf{B} = \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla) B^2/2 + B^2 \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$, the Lorentz force can be written with all terms orthogonal to $\mathbf{B}$:

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = -\nabla_\perp \frac{B^2}{8\pi} + \frac{B^2}{4\pi}\,\kappa, \tag{19}$$

where $\nabla_\perp \equiv \nabla - \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla)$. The **magnetic tension** term $\propto \kappa$ tends to straighten curved field lines; the **magnetic pressure** $p_B \equiv B^2/(8\pi)$ pushes from regions of high B^2 toward regions of low B^2.

The ratio of plasma pressure to magnetic pressure is the **plasma beta**:

$$\beta \equiv \frac{p}{B^2/8\pi}. \tag{20}$$

When $\beta \ll 1$ the magnetic field dominates (e.g. solar corona, tokamaks); when $\beta \gg 1$ plasma pressure dominates (e.g. stellar interiors); when $\beta \sim 1$ both forces matter (e.g. solar chromosphere, parts of the solar wind).

Energy, induction, and constraints

Adiabatic energy equation

So far we have one continuity equation for ρ, three momentum components for $\mathbf{V}$, and the magnetic field $\mathbf{B}$ (three components, with $\nabla \cdot \mathbf{B} = 0$). That leaves **pressure** p as an extra unknown with no equation of its own. In a full plasma treatment one would write a **total energy equation** for the internal energy (or temperature) of the fluid, including compression work, heat conduction, viscous heating, radiation, and resistive dissipation. Ideal MHD drops all of those transport and dissipation terms. What remains is the statement that each fluid element evolves **adiabatically**: no heat enters or leaves the element as it moves.

For an ideal gas with ratio of specific heats $\gamma \equiv C_p/C_v$ (often $\gamma = 5/3$ for a monatomic plasma), a familiar thermodynamic identity is $p \propto \rho^\gamma$ at constant entropy. More precisely, the quantity p/ρ^γ is proportional to the entropy per unit mass, so requiring adiabatic flow along a fluid trajectory is

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{21}$$

where $d/dt \equiv \partial/\partial t + \mathbf{V} \cdot \nabla$ is the material derivative introduced in 8. In words: pressure and density change together as a fluid element is compressed or rarefied, with no external heating. This is the **closure relation** that completes the ideal MHD system.

Applying the material derivative and using the continuity equation 7, 21 can be rewritten in **Eulerian** form as

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p = -\gamma \rho \, \nabla \cdot \mathbf{V}, \tag{22}$$

where the right-hand side is the adiabatic compression term: when the flow converges ($\nabla \cdot \mathbf{V} < 0$), pressure rises; when it diverges, pressure falls. Thermal conduction, radiation, and resistive or viscous heating are all neglected, so this closure is best viewed as a useful approximation (for example in linear MHD waves) rather than a precise description of every astrophysical plasma.

> Same idea as cosmological fluid closure?
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> The adiabatic MHD relation is the same **thermodynamic** closure that appears whenever one ties p to ρ without evolving temperature separately. In cosmology, a perfect-fluid source for the Friedmann equations also needs such a link. The standard parameterization is the **equation-of-state parameter** $w \equiv p/(\rho c^2)$: non-relativistic matter has $w \approx 0$ (pressure negligible in the background), radiation has $w = 1/3$, and dark energy in ΛCDM has $w \approx -1$. Ideal MHD keeps the **non-relativistic ideal-gas** branch: a single polytropic law $p \propto \rho^\gamma$ with $\gamma = C_p/C_v$.
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> The physics is the same adiabatic picture. Compress a comoving volume in an expanding universe, or compress a plasma element in MHD, and p rises with ρ with no heat added. For non-relativistic adiabatic gas one finds the familiar scaling $T \propto a^{-3(\gamma-1)}$ in cosmology and, in MHD, 22 with $\nabla \cdot \mathbf{V}$ as the compression term.
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> The packaging differs. Cosmology uses w because radiation and dark energy are not simple $p \propto \rho^\gamma$ polytropes. Perturbation theory also uses **adiabatic initial conditions** ($\delta p/p = c_s^2 \, \delta\rho/\rho$), a related but more specialized meaning of "adiabatic." MHD instead follows a plasma parcel with velocity $\mathbf{V}$ and magnetic field $\mathbf{B}$ in flat space. Still, 21 is the same entropy-conservation closure you would invoke for an adiabatic gas component before specializing to $w = 0$, $1/3$, or -1.

Faraday's law and ideal Ohm's law

Faraday's law is unchanged from Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c \, \nabla \times \mathbf{E}. \tag{23}$$

The electric field in the rest frame of a conductor moving at velocity $\mathbf{V}$ is, to lowest order in

V^2/c^2,

$$\mathbf{E}' \approx \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}. \tag{24}$$

Setting $\mathbf{E}' = 0$ for a perfect conductor gives the **ideal Ohm's law**:

$$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0, \tag{25}$$

which is Galilean invariant at this order. Resistivity, the Hall effect, electron inertia, and ambipolar diffusion are neglected.

Substituting 25 into 23 yields the **induction equation**:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}). \tag{26}$$

The frozen-in theorem

Equation 26 is the **frozen-in** condition of ideal MHD. To see what it means, return to ideal Ohm's law 25: in the frame moving with the plasma velocity $\mathbf{V}$, the electric field vanishes ($\mathbf{E}' = 0$). A **perfect conductor** cannot maintain an electric field inside it; any tendency for the magnetic field to change would drive a current that cancels $\mathbf{E}'$. Real plasmas have small but nonzero resistivity, so this is an approximation, but a powerful one on fast MHD time scales.

Substituting 25 into Faraday's law eliminates $\mathbf{E}$ and shows that $\mathbf{B}$ is carried by the flow. The induction equation says that $\mathbf{B}$ is advected, stretched, and rotated by $\mathbf{V}$, much like a line of dye in a moving fluid. **Magnetic field lines move with the plasma**; they do not slip through it. If a blob of plasma is tied to a field line, they go together. Stretch the plasma and the field line stretches; compress a flux tube and the field strength increases (flux conservation in a narrowing tube).

Magnetic topology is the pattern of how field lines link and thread through space: which lines encircle which, how many times they wrap, and so on. In ideal MHD that topology **cannot change**: field lines may be deformed but not cut and reconnected. Processes such as **magnetic reconnection** (as in solar flares or tokamak disruptions) require non-ideal effects (resistivity, the Hall term, or electron inertia) that allow field lines to break and join across a thin layer.

In short: ideal MHD treats the plasma as a perfect conductor, and the induction equation then implies that magnetic field lines are **frozen into** the fluid. That is what we mean when we say the field and plasma are frozen together.

Ampère's law and the divergence constraint

At low frequencies we neglect displacement current in Ampère's law:

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}. \tag{27}$$

There is no explicit time dependence, so $\nabla \cdot \mathbf{J} = 0$, consistent with quasineutrality. Recall that $\mathbf{J}$ also represents the relative drift between ions and electrons (10).

Gauss's law for magnetism requires

$$\nabla \cdot \mathbf{B} = 0. \tag{28}$$

Magnetic monopoles do not exist; $\mathbf{B}$ is solenoidal. If $\mathbf{B}$ is initially divergence-free, it remains so: taking the divergence of 23 gives $\partial(\nabla \cdot \mathbf{B})/\partial t = 0$ because the divergence of a curl vanishes identically.

Writing $\mathbf{B} = \nabla \times \mathbf{A}$ automatically satisfies 28. Gauge freedom allows $\mathbf{A}' = \mathbf{A} + \nabla\phi$ without changing $\mathbf{B}$.

Summary

Ideal MHD in cgs units couples hydrodynamics with the low-frequency, long-wavelength limit of Maxwell's equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \tag{29}$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p, \tag{30}$$

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0, \tag{31}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0, \tag{32}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{33}$$

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \tag{34}$$

Extensions to resistive MHD, the Hall effect, and two-fluid models are often needed for reconnection, turbulence, and laboratory plasmas; ideal MHD remains a strong predictor of macroscopic stability.

References

Useful sources on MHD include Boyd & Sanderson [2], Kulsrud [3], Schnack [4], and Priest [5]. The present condensation follows Murphy's Ay253 lecture notes [1].

[1] N. Murphy, "Ideal magnetohydrodynamics." Astronomy 253: Plasma Astrophysics, Harvard University, 2016.

[2] T. J. M. Boyd and J. J. Sanderson, *The physics of plasmas.* Cambridge University Press, 2003.

[3] R. M. Kulsrud, *Plasma physics for astrophysics.* Princeton University Press, 2005.

[4] D. D. Schnack, *Lectures in magnetohydrodynamics.* Springer, 2009.

[5] E. Priest, *Magnetohydrodynamics of the sun.* Cambridge University Press, 2014.