Parametric Amplifier

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This work presents a theoretical analysis of non-degenerate parametric amplifiers, which achieve signal amplification through time-varying reactive elements without introducing thermal noise. We derive the fundamental current-voltage relationships for parametric circuits and systematically eliminate variables to obtain the signal admittance expression. This result demonstrates how parametric coupling creates effective negative conductance proportional to pump power, enabling amplification when the negative conductance exceeds the circuit's passive losses.

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Today we will be discussing the non-degenerate parametric amplifier. I will be borrowing the derivation from [1], [2], and [3].

Devices with nonlinear susceptance can be used to create parametric amplifiers and oscillators. A well known example is the reverse-biased pn junction, which has nonlinear charge-voltage characteristics arising from voltage-dependent capacitance. Such a nonlinear behavior results in frequency mixing among the following components:

- signal,
- idler,
- pump.

The energy is transferred from the pump wave to the weaker signal and idler waves. This is the operating principle of the parametric amplifier.

Circuit equivalent of a parametric amplifier is shown in Figure 1 or in its alternative form in Figure 2.

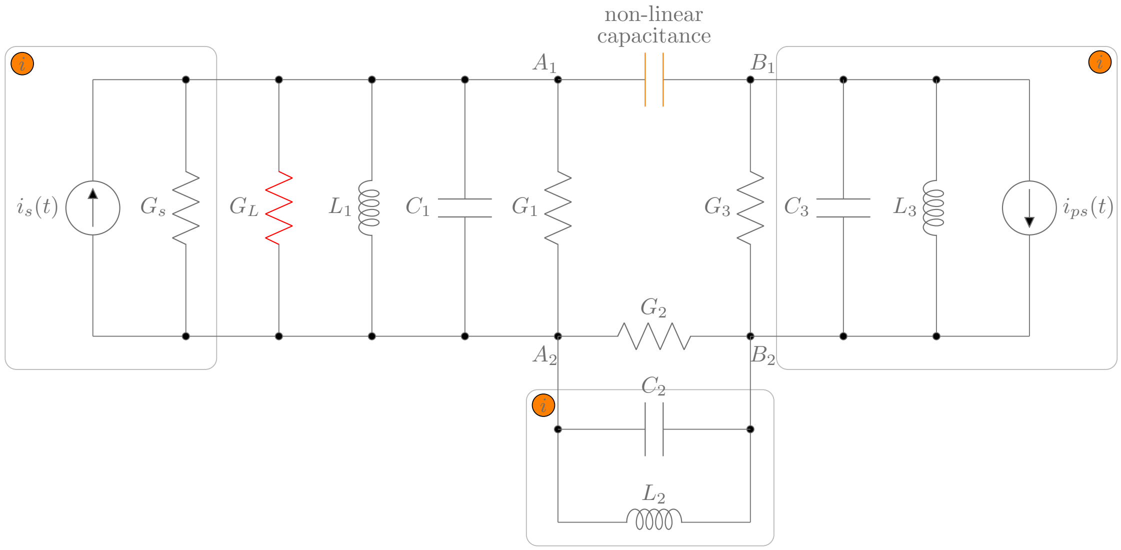

Figure 1: Non-degenerate parametric amplifier[2]. Hover over the orange circuit elements to see more information.

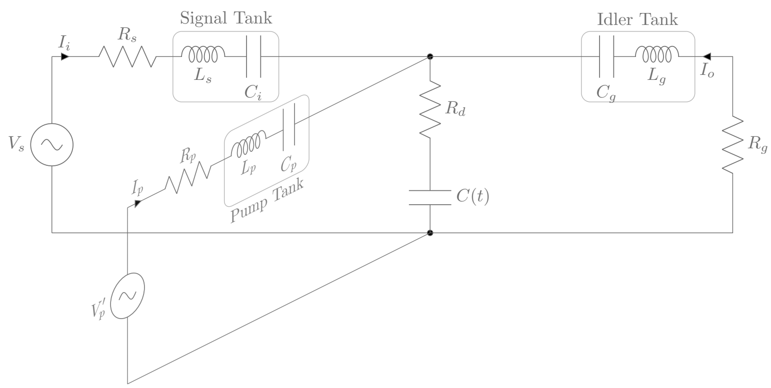

Figure 2: An alternative representation of the circuit.

Nonlinear Capacitance

Consider a capacitor which has a voltage-dependent capacitance. Such dependence naturally arises from the pn junction diode.

$$\mathcal{C} = C(1 + \alpha v), \tag{1}$$

where C is the capacitance at zero voltage and α represents the linear dependence of the capacitance on the voltage. The charge on the capacitor is then

$$q(t) = C\left(1 + \alpha v(t)\right)v(t) = Cv(t) + av^2(t), \tag{2}$$

where $a = \alpha C$. The current through the capacitor is

$$i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt} + 2av(t)\frac{dv(t)}{dt}. \tag{3}$$

We will assign the angular frequencies ω_1, ω_2, and ω_3 for the signal, idler and pump waves, respectively. In Figure 1 we have chosen the sign of the currents such that the total voltage across the nonlinear capacitance is the sum of the voltages across the signal, idler and pump circuits. The voltage across the nonlinear capacitor is given by

$$v(t) = v_1(t) + v_2(t) + v_3(t) \tag{4}$$
$$= V_1\cos(\omega_1 t + \phi_1) + V_2\cos(\omega_2 t + \phi_2) + V_3\cos(\omega_3 t + \phi_3) \tag{5}$$

The relations between ω's are easier to see in the alternative form of the circuit in Figure 2. Imagine that the idler circuit is disconnected. The capacitor will have a tank circuit on the left with the resonant frequency ω_1 and the signal circuit will have a tank circuit on the right with the resonant frequency ω_3. The idler circuit will be tuned to beat at the difference frequency $\omega_2 = \omega_3 - \omega_1$, or equivalently:

$$\omega_3 = \omega_1 + \omega_2 \tag{6}$$

Now let's define the individual frequencies of the signals. It is done with the typical trick of disabling all but one of the voltage/current sources. One other ingredient is the observation that a parallel $L-C$ tank circuit becomes a short circuit at frequencies far separated from the resonant frequency. As we look from the signal side into the circuit in Figure 1 at a frequency of ω_1, the idler and pump circuits become short circuits making C and C_1 appear in parallel. The same argument applies to the idler side and the pump side of the circuit.

Therefore the angular frequencies in Eq. 5 satisfy

$$\omega_k = \frac{1}{\sqrt{L_k(C_k + C)}}, \tag{7}$$

where $k = 1, 2, 3$.

Current-Voltage Relations

Now we take the voltage expression Eq. 5 and substitute it in Eq. 3 and reorganize the terms:

$$\begin{aligned} i(t) &= -\omega_1 C V_1 \sin(\omega_1 t + \phi_1) - \omega_1 a V_2 V_3 \sin(\omega_1 t + \phi_3 - \phi_2) \\ &\quad -\omega_2 C V_2 \sin(\omega_2 t + \phi_2) - \omega_2 a V_1 V_3 \sin(\omega_2 t + \phi_3 - \phi_1) \\ &\quad -\omega_3 C V_3 \sin(\omega_3 t + \phi_3) - \omega_3 a V_1 V_2 \sin(\omega_3 t + \phi_1 + \phi_2). \end{aligned} \tag{8}$$

Let's label the terms in the above equation as follows:

$$i_1(t) = -\omega_1 C V_1 \sin(\omega_1 t + \phi_1) - \omega_1 a V_2 V_3 \sin(\omega_1 t + \phi_3 - \phi_2) \tag{9}$$

$$i_2(t) = -\omega_2 C V_2 \sin(\omega_2 t + \phi_2) - \omega_2 a V_1 V_3 \sin(\omega_2 t + \phi_3 - \phi_1) \tag{10}$$

$$i_3(t) = -\omega_3 C V_3 \sin(\omega_3 t + \phi_3) - \omega_3 a V_1 V_2 \sin(\omega_3 t + \phi_1 + \phi_2) \tag{11}$$

Equations 9–11 combine to give the total current:

$$i(t) = i_1(t) + i_2(t) + i_3(t) \tag{12}$$

We want to convert Eqs. 9–11 to the form of the current-voltage relations using the components of the voltage in Eq. 5. We will use the following trigonometric identity:

$$\begin{aligned} \sin(\omega_1 t + \phi_3 - \phi_2) &= \sin(\omega_1 t + \phi_1 + \phi_3 - \phi_2 - \phi_1) \\ &= \sin(\omega_1 t + \phi_1)\cos(\phi_3 - \phi_2 - \phi_1) + \cos(\omega_1 t + \phi_1)\sin(\phi_3 - \phi_2 - \phi_1) \\ &= \frac{1}{V_1 \omega_1}\left[\dot{v}_1(t)\cos(\phi_3 - \phi_2 - \phi_1) + \omega_1 v_1(t)\sin(\phi_3 - \phi_2 - \phi_1)\right] \end{aligned} \tag{13}$$

This conversion gives:

$$\begin{aligned} i_1(t) &= C\dot{v}_1(t) + \frac{aV_1V_3}{V_1}\left[\cos(\phi_3 - \phi_2 - \phi_1)\dot{v}_1(t) - \omega_1 v_1(t)\sin(\phi_3 - \phi_2 - \phi_1)\right] && (14) \\ i_2(t) &= C\dot{v}_2(t) + \frac{aV_3}{V_2}\left[\cos(\phi_3 - \phi_2 - \phi_1)\dot{v}_2(t) - \omega_2 v_2(t)\sin(\phi_3 - \phi_2 - \phi_1)\right] && (15) \\ i_3(t) &= C\dot{v}_3(t) + \frac{aV_1V_2}{V_3}\left[\cos(\phi_3 - \phi_2 - \phi_1)\dot{v}_3(t) + \omega_3 v_3(t)\sin(\phi_3 - \phi_2 - \phi_1)\right] && (16) \end{aligned}$$

Admittance

Moving to the frequency domain, Eqs. 14–16, we get the admittances Y_k (k = 1,2,3) as we look into the circuit from point $A_1 - A_2$, $A_2 - B_2$, and $B_1 - B_2$, respectively:

$$
\begin{aligned}
Y_1 &= \frac{I_1(i\omega)}{V_1(i\omega)} = i\omega_1 C + i\omega_1 a \frac{V_2 V_3}{V_1} \exp[i(\phi_3 - \phi_2 - \phi_1)] && (17)\\
Y_2 &= \frac{I_2(i\omega_2)}{V_2(i\omega_2)} = i\omega_2 C + i\omega_2 a \frac{V_1 V_3}{V_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] && (18)\\
Y_3 &= \frac{I_3(i\omega)}{V_3(i\omega)} = i\omega_3 C + i\omega_3 a \frac{V_1 V_2}{V_3} \exp[-i(\phi_3 - \phi_2 - \phi_1)] && (19)
\end{aligned}
$$

To see the total admittance from the point of view of the signal, for example, we need to add the parallel admittances of the G_s, G_L, L_1, C_1, G_1. It is a bit tricky since one may be inclined to think that the $L_1 - C_1$ tank will have 0 admittance at ω_1. However, we have shown in Eq. 7 that the resonance frequency is shifted by C. Let's calculate the admittance of the $L_1 - C_1$ tank at ω_1:

$$
Y_{L_1 - C_1} = \frac{1}{i\omega_1 L_1} + i\omega_1 C_1 = \frac{1 - \omega_1^2 L_1 C_1}{i\omega_1 L_1} = \frac{\omega_1^2 L_1 C_1}{i\omega_1 L_1} = -i\omega_1 C_1, \tag{20}
$$

which neatly cancels the first term in Eq. 17. All there is left to do is to add the parallel admittances of the G's, $G_T = G_s + G_L + G_1$ for the signal circuit.

The current-voltage relations for the three circuits are given by

$$
\begin{aligned}
I_s(i\omega) &= \left\{ G_T + i\omega_1 a \frac{V_2 V_3}{V_1} \exp[i(\phi_3 - \phi_2 - \phi_1)] \right\} V_1(i\omega) && (21)\\
0 &= \left\{ G_2 + i\omega_2 a \frac{V_1 V_3}{V_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] \right\} V_2(i\omega) && (22)\\
I_P(i\omega) &= \left\{ G_3 + i\omega_3 a \frac{V_1 V_2}{V_3} \exp[-i(\phi_3 - \phi_2 - \phi_1)] \right\} V_3(i\omega) && (23)
\end{aligned}
$$

Here $I_s(i\omega)$ and $I_P(i\omega)$ are the Fourier transforms of the input signal and pump currents, respectively.

By eliminating V_2 and V_3 from Eq. 21 using Eqs. 22 and 23, we obtain the admittance of the signal circuit.

Step-by-step derivation

Solve for V_2 from 22

$$V_2 = \frac{i\omega_2 a V_1 V_3}{G_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] \tag{24}$$

Substitute V_2 into 23
Substituting 24 into 23:

$$\begin{aligned} I_P(i\omega) &= \left\{ G_3 + i\omega_3 a \frac{V_1}{V_3} \cdot \frac{i\omega_2 a V_1 V_3}{G_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] \exp[-i(\phi_3 - \phi_2 - \phi_1)] \right\} V_3(i\omega) \\ &= \left\{ G_3 + i\omega_3 a \cdot \frac{i\omega_2 a V_1^2}{G_2} \right\} V_3(i\omega) \end{aligned} \tag{25}$$

Solve for V_3 from 25:

$$V_3 = \frac{I_P(i\omega)}{G_3 - \frac{\omega_2 \omega_3 a^2 V_1^2}{G_2}} \tag{26}$$

Using 24 and 26 we find $V_2 V_3$ product

$$\begin{aligned} V_2 V_3 &= \frac{i\omega_2 a V_1 V_3}{G_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] \cdot V_3 \\ &= \frac{i\omega_2 a V_1}{G_2} \exp[i(\phi_3 - \phi_2 - \phi_1)] \cdot \frac{I_P^2(i\omega)}{\left(G_3 - \frac{\omega_2 \omega_3 a^2 V_1^2}{G_2} \right)^2} \end{aligned} \tag{27}$$

Calculate Signal Admittance
The signal admittance is $Y_s = I_s(i\omega)/V_1(i\omega)$. From 21:

$$Y_s = G_T + i\omega_1 a \frac{V_2 V_3}{V_1} \exp[i(\phi_3 - \phi_2 - \phi_1)] \tag{28}$$

After substituting the expression for $V_2 V_3$ and performing algebraic manipulations involving the complex exponentials and denominators, we obtain:

$$Y_s = G_T - G = G_T - \frac{\omega_1 \omega_2 a^2}{G_2 G_3} \frac{|I_P(i\omega)|^2}{\left| 1 + \frac{\omega_2 \omega_3}{G_2 G_3} a^2 V_1^2 \right|^2} \tag{29}$$

The negative conductance emerges as a result of the nonlinear capacitance driven by the pump wave at ω_3. If V_1 satisfies the condition,

$$\frac{\omega_2\omega_3}{G_2G_3}a^2V_1^2 < 1 \tag{30}$$

the negative conductance is independent of the signal input and the linear parametric amplification is realized.

Power Gain

The ratio of the power delivered to the load G_L to the input power to the source G_s is the gain in power.

$$\mathcal{G} = \frac{G_LV_1^2}{|I_s|^2/4G_s} = \frac{4G_sG_L}{|Y_s|^2}. \tag{31}$$

[1] H. Heffner and G. Wade, "Gain, bandwidth and noise characteristic of the variable parameter amplifier," *Journal of Applied Physics*, vol. 29, pp. 1321–1331, 1958, doi: 10.1063/1.1723440.

[2] National Institute of Informatics, "Parametric amplifiers and oscillator." National Institute of Informatics, Japan, 2024 [Online]. Available: https://www.nii.ac.jp/qis/first-quantum/e/forStudents/lecture/pdf/noise/chapter11.pdf. [Accessed: 2024]

[3] W. J. Robertson, "Parametric amplifier design," PhD thesis, The Ohio State University, 1959 [Online]. Available: http://rave.ohiolink.edu/etdc/view?acc_num=osu1400145234