Inductance of a Wire Pair

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This article examines the mutual inductance between parallel wire segments, a fundamental configuration in electrical circuits and transmission lines. Building upon our previous analysis of single-wire self-inductance, we derive the magnetic coupling between current-carrying conductors using the Biot-Savart law. We address the mathematical challenges of finite-length conductors and present a complete solution that includes both the self-inductance of each wire and their mutual coupling.

blog: https://tetraquark.vercel.app/posts/wirePairInductance/ email: quarktetra@gmail.com

Magnetic Field of a Wire Segment

I have discussed the self-inductance of a wire segment in a previous post. Now I will discuss the mutual inductance of two wire segments. For completeness, I will reproduce parts of the self-inductance calculations.

Figure 1 shows a wire segment of length L carrying a current I and located at the origin, and its pair carrying the return current I and located at $z = d$.

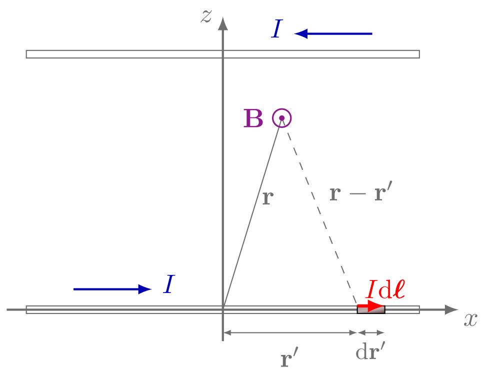

Figure 1: A segment of a wire carrying current I and its pair carrying the return current at $z = d$.

The magnetic field at an arbitrary point $\mathbf{r}$ created by a current distribution $\mathbf{J}(\mathbf{r}')$ is given by the Biot-Savart law[1]:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\,\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^3}, \tag{1}$$

$$\mathbf{J}(\mathbf{r}') = \frac{I}{\pi\rho_0^2}\Theta(\rho_0-\rho')\hat{\mathbf{x}}, \tag{2}$$

Due to the rotational symmetry of the set up, we can compute the magnetic field at $y = 0$ and then rotate it to the desired angle.

$$\mathbf{r}-\mathbf{r}' = z\,\hat{\mathbf{z}} + (x-x')\,\hat{\mathbf{x}}, \tag{3}$$

Let us first consider the magnetic field outside the wire., i.e., $z > \rho_0$.

$$\begin{aligned}
\mathbf{B}(x,z) &= \frac{\mu_0}{4\pi}I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\frac{\hat{\mathbf{x}}\times(z\,\hat{\mathbf{z}}+(x-x')\,\hat{\mathbf{x}})}{(z^2+(x-x')^2)^{3/2}} = z\,\hat{\mathbf{y}}\frac{\mu_0}{4\pi}I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\frac{1}{(z^2+(x-x')^2)^{3/2}} \\
&= \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\int_{-\frac{L/2-x}{z}}^{\frac{L/2+x}{z}} du\,\frac{1}{(1+u^2)^{3/2}} = \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\left[\frac{u}{\sqrt{1+u^2}}\right]_{-\frac{L/2-x}{z}}^{\frac{L/2+x}{z}} \\
&= \hat{\mathbf{y}}\frac{\mu_0}{4\pi z}I\left[\frac{x+L/2}{\sqrt{z^2+(x+L/2)^2}} - \frac{x-L/2}{\sqrt{z^2+(x-L/2)^2}}\right].
\end{aligned} \tag{4}$$

The Linked-Flux

The concept of linked-flux is a bit ambiguous for segments of wires which are not closed. To address this issue, imagine the the top wire is actually an arc of a circle of radius R and length L. With this picture, it is more clear that the flux created by the lower wire and linked to the upper wire is in the domain $z > d$. We can recover the straight wire result by taking the limit $R \to \infty$.

On the x axis, we will want to integrate over x to compute the flux, however, we can't really integrate from $-\infty$ to ∞ because this will enclose other wires supplying current to the wire we are considering. Therefore, we will have to integrate over a finite interval, say $[-L/2, L/2]$. This methodology is consistent with the other method of computing the self-inductance of a wire segment as proposed by Neumann. [2] [3] [4]

The differential flux going through a strip of width dx at z is given by

$$d\Phi(z) = dz \int_{-L/2}^{L/2} dx \mathbf{B}(x,z) = dz \frac{\mu_0 I}{4\pi z} \left[\sqrt{z^2 + (x+L/2)^2} - \sqrt{z^2 + (x-L/2)^2} \right]_{-L/2}^{L/2} = dz \frac{\mu_0 I}{2\pi z} \left[\sqrt{z^2 + L^2} - z \right] \quad (5)$$

Finally, we integrate over z to get the total flux:

$$\Phi = \int_{\rho_0}^{\infty} d\Phi(z) = \frac{\mu_0 I}{2\pi} \int_{\rho_0}^{\infty} \frac{dz}{z} \left[\sqrt{z^2 + L^2} - z \right] = \frac{\mu_0 I}{2\pi} \int_{\rho_0}^{\infty} dz \left[\frac{\sqrt{z^2 + L^2}}{z} - 1 \right] \quad (6)$$

Let's do the harder integral first with the substitution $z = L \sinh t$:

$$\begin{aligned} I &= \int dz \frac{\sqrt{z^2 + L^2}}{z} = L \int dt \frac{\cosh^2 t}{\sinh t} = L \int dt \left[\frac{1}{\sinh t} + \sinh t \right] = L \int dt \frac{2}{e^t - e^{-t}} + L \cosh t \\ &= L \int d(e^t) \frac{2}{e^{2t} - 1} + L \cosh t = L \left[\ln \left(\frac{e^t + 1}{e^t - 1} \right) + \sinh t \right] = L \left[\ln \left(\tanh \frac{t}{2} \right) + \cosh t \right] \quad (7) \end{aligned}$$

Now use $\tanh \frac{t}{2} = \frac{\sinh t}{1 + \cosh t}$ to get:

$$I = L \ln \left(\frac{z}{L + \sqrt{z^2 + L^2}} \right) + \sqrt{z^2 + L^2}. \quad (8)$$

Putting it all together with the limits we get:

$$\begin{aligned} \Phi &= \frac{\mu_0 I}{2\pi} \left[L \ln \left(\frac{z}{L + \sqrt{z^2 + L^2}} \right) + \sqrt{z^2 + L^2} - z \right]_d^{\infty} \\ &= \frac{\mu_0 I}{2\pi} \left[-L \ln \left(\frac{d}{L + \sqrt{d^2 + L^2}} \right) - \sqrt{d^2 + L^2} + d \right] \simeq \frac{\mu_0 I L}{2\pi} \left[\ln \left(\frac{2L}{d} \right) - 1 \right], \quad (9) \end{aligned}$$

where we assumed $d \ll L$. Using the relation $\mathcal{L} = I\Phi$ we get:

$$\mathcal{M} = \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{d}\right) - 1\right], \tag{10}$$

where $\mathcal{M}$ is the mutual inductance.

If these two wires are part of a closed loop, the total inductance is given by the sum of the self-inductance of each wire plus the mutual inductance between them.

$$\mathcal{L}_{total} = 2\mathcal{L} - 2\mathcal{M}, \tag{11}$$

where $\mathcal{L}$ is the self-inductance of each wire.We calculated the self-inductance of a wire segment in a previous post:

$$\mathcal{L} = \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{\rho_0}\right) - 1 + \frac{\mu}{4\mu_0}\right]. \tag{12}$$

Therefore, the total inductance is given by:

$$\mathcal{L}_{total} = 2\mathcal{L} - 2\mathcal{M} = \frac{\mu_0 L}{\pi}\left[\ln\left(\frac{d}{\rho_0}\right) + \frac{\mu}{4\mu_0}\right]. \tag{13}$$

Although it was instructive to calculate the inductance of the wire segments in terms of its self and mutual inductance components, we could have calculated the total inductance directly by integrating the net flux:

$$\Phi = \frac{\mu_0 I}{\pi}\int_{\rho_0}^{d-\rho_0}\frac{dz}{z}\left[\sqrt{z^2+L^2} - z\right] \simeq \frac{\mu_0 I}{\pi}\ln\left(\frac{d}{\rho_0}\right) \tag{14}$$

, which results in the same total inductance as before.

[1] D. J. Griffiths, *Introduction to electrodynamics*. Pearson, 2013.

[2] F. E. Neumann, "Allgemeine gesetze der inducirten elektrischen ströme," *Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin*, pp. 1–87, 1847.

[3] E. B. Rosa, "The self and mutual inductances of linear conductors," *Bulletin of the Bureau of Standards*, vol. 4, no. 2, pp. 301–344, 1907.

[4] R. Dengler, "Self inductance of a wire loop as a curve integral." 2013 [Online]. Available: https://arxiv.org/abs/1204.1486