Inductance of a Wire Pair with Neumann's Method

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This article presents a detailed derivation of the mutual inductance between two parallel wire segments using Neumann's method. Starting from the fundamental electromagnetic energy expression involving the vector potential and current density, we evaluate the mutual inductance through direct integration. The analysis assumes thin wire approximation and provides the final result in terms of the wire length and separation distance. This approach offers an alternative perspective to the more commonly used flux-based calculations, while arriving at the same well-known logarithmic dependence on the geometric parameters.

blog: https://tetraquark.vercel.app/posts/wirePairInductanceNeumann/ email: quarktetra@gmail.com

Energy in the field

I have discussed the self-inductance of a wire segment in a previous post and the mutual inductance of two wire segments in a previous post. Now I will revisit the mutual inductance of two wire segments using Neumann's method.

Figure 1 shows a wire segment of length L carrying a current I and located at the origin, and its pair carrying the return current I and located at $z = d$.

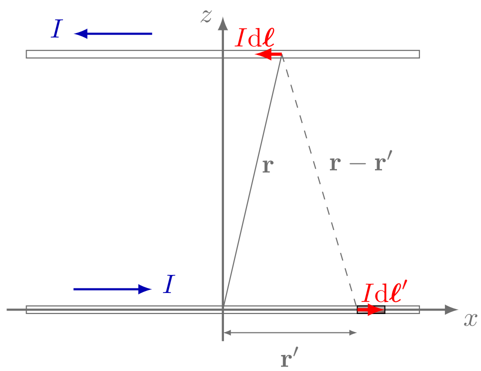

Figure 1: A segment of a wire carrying current I and its pair carrying the return current at $z = d$.

The energy stored in the field due to the interaction of the two wires is roughly given by $\mathbf{A}\cdot\mathbf{J}$. The origin of this expression can be traced back to gauging the quantum description of electrons with the vector potential $\mathbf{A}$ which couples A to the current density $\mathbf{J}$.

$$\mathcal{W} = \frac{1}{2}\int d^3\mathbf{r}\mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{8\pi}\int d^3\mathbf{r}d^3\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}')\cdot\mathbf{J}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

where the current densities $\mathbf{J}$ are concentrated in the wire segments:

$$\mathbf{J}(\mathbf{r}') = \frac{I}{\pi\rho_0^2}\Theta(\rho_0-\rho')\hat{\mathbf{x}}. \tag{2}$$

The integrals in Eq. 1 collapses to line integrals over the wire segments:

$$\mathcal{W} = \frac{\mu_0 I^2}{8\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx' \int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{\sqrt{d^2+(x-x')^2}}. \tag{3}$$

Let us evaluate the integral over x first by defining $(x-x')/d \equiv \sinh t$:

$$\begin{aligned}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{\sqrt{d^2+(x-x')^2}} &= \int_{-\sinh^{-1}(x'/d+\frac{L}{2d})}^{\sinh^{-1}(\frac{L}{2d}-x'/d)} d\sinh t\frac{1}{\sqrt{1+\sinh^2 t}}\\ &= \sinh^{-1}\left(x'/d+\frac{L}{2d}\right)-\sinh^{-1}\left(x'/d-\frac{L}{2d}\right).\end{aligned} \tag{4}$$

The next step will require us to integrate $\sinh^{-1}(u)$. Let's define $v=\sinh(u)$, and derive the following identity:

$$\begin{aligned}\int du\sinh^{-1}(u) &= \int dv\, v\cosh v = \int d\,(v\sinh v) - \int d\cosh v = v\sinh v-\cosh v\\ &= \sinh^{-1}(u)u-\sqrt{1+u^2}\end{aligned} \tag{5}$$

Furthermore, we can use the identity $\sinh^{-1} u = \ln\left(u + \sqrt{1+u^2}\right)$. Putting it all together we get:

$$\mathcal{W} \quad = \quad \frac{\mu_0 I^2}{4\pi}\left[-L\ln\left(\frac{d}{L+\sqrt{d^2+L^2}}\right) - \sqrt{d^2+L^2} + d\right] \simeq \frac{1}{2}\left(\frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{d}\right) - 1\right]\right) I^2, \tag{6}$$

where we assumed $d \ll L$. Using the relation $\mathcal{W} = \frac{1}{2}\mathcal{M}I^2$ we get:

$$\mathcal{M} \quad = \quad \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{2L}{d}\right) - 1\right], \tag{7}$$

where $\mathcal{M}$ is the mutual inductance.